CLASSICAL EQUIVALENCE AND QUANTUM EQUIVALENCE OF MAGNETIC FIELDS ON FLAT TORI

CAROLYN GORDON, WILLIAM KIRWIN, DOROTHEE SCHUETH, AND DAVID WEBB

ABSTRACT. Let M be a real $2m$-torus equipped with a translation-invariant metric h and a translation-invariant symplectic form ω; the latter we interpret as a magnetic field on M. The Hamiltonian flow of half the norm-squared function induced by h on T^*M (the "kinetic energy") with respect to the twisted symplectic form $\omega_{T^*M} + \pi^*\omega$ describes the trajectories of a particle moving on M under the influence of the magnetic field ω. If $[\omega]$ is an integral cohomology class, then we can study the geometric quantization of the symplectic manifold $(T^*M, \omega_{T^*M} + \pi^*\omega)$ with the kinetic energy Hamiltonian. We say that the quantizations of two such tori (M_1, h_1, ω_1) and (M_2, h_2, ω_2) are *quantum equivalent* if their quantum spectra, i.e., the spectra of the associated quantum Hamiltonian operators, coincide; these quantum Hamiltonian operators are proportional to the h_j-induced bundle Laplacians on powers of the Hermitian line bundle on M with Chern class $[\omega]$.

In this paper, we construct continuous families $\{(M, h_t)\}_t$ of mutually nonisospectral flat tori (M, h_t), each endowed with a translation-invariant symplectic structure ω, such that the associated classical Hamiltonian systems are pairwise equivalent. If ω represents an integer cohomology class, then the (M, h_t, ω) also have the same quantum spectra. We show moreover that for any translation-invariant metric h and any translation-invariant symplectic structure ω on M that represents an integer cohomology class, the associated quantum spectrum determines whether (M, h, ω) is Kähler, and that all translation-invariant Kähler structures (h, ω) of given volume on M have the same quantum spectra. Finally, we construct pairs of magnetic fields (M, h, ω_1), (M, h, ω_2) having the same quantum spectra but nonsymplectomorphic classical phase spaces. In some of these examples the pairs consist of Kähler manifolds.

1. INTRODUCTION

Consider an even-dimensional torus $M = \mathbb{Z}^{2m}\backslash\mathbb{R}^{2m}$. To each translation-invariant closed 2-form ω and translation-invariant (i.e., flat) Riemannian metric h on M, associate a Hamiltonian system (T^*M, Ω, H). Here Ω is the symplectic form on T^*M given by $\Omega = \omega_0 + \pi^*\omega$, where ω_0 is the Liouville form, and $\pi : T^*M \to M$ is the projection. The Hamiltonian function H is given by $H(q, \xi) = \frac{1}{2}h_q(\xi, \xi)$. In case $\omega = 0$, the Hamiltonian system gives the classical geodesic flow. A nontrivial closed 2-form ω may be viewed as a magnetic field on M, and the Hamiltonian system describes the dynamics of a charged particle moving in the magnetic field. We will say that (M, h_1, ω_1) and (M, h_2, ω_2) are *classically equivalent* if the associated

2010 *Mathematics Subject Classification.* Primary 58J53; Secondary 53C30, 53C20.

The first author and last author were supported in part by NSF Grants DMS 0605247 and DMS 0906169. The third author was partially supported by DFG Sonderforschungsbereich 647. Moreover, the third author thanks Dartmouth College and its Harris visiting program for great hospitality and support.

Hamiltonian systems are equivalent, i.e., if there is a symplectomorphism of cotangent bundles intertwining the Hamiltonian functions.

If, moreover, ω represents an integer cohomology class, then there exists a Hermitian complex line bundle L with Chern class $[\omega]$. Choose a Hermitian connection ∇ with curvature $-2\pi i\omega$. The connection gives rise to a Hermitian connection, also denoted ∇, on each tensor power $L^{\otimes k}$, i.e., on the line bundles with Chern class $k\omega$, $k \in \mathbb{Z}^+$.

According to the procedure of geometric quantization (specifically, with respect to the vertical polarization on the cotangent bundle in the presence of the metaplectic correction), the quantum Hilbert space at level $\hbar = 1/k$ ($k \in \mathbb{Z}^+$) associated to $(T^*M, \omega_0 + \pi^*\omega)$ is the L^2-space of square integrable sections of $L^{\otimes k}$. The quantum Hamiltonian associated to the classical Hamiltonian H is the operator $\widehat{H}_k = \frac{\hbar^2}{2}\Delta$, where $\Delta = -\,\textit{trace}(\nabla^2)$. (See [9], and note that the scalar curvature term appearing there is zero in our case. Also see Section 2 of [4] for a brief outline of geometric quantization.)

For technical reasons, we will always assume that ω is nondegenerate, i.e., that it is a symplectic structure on M. Of course, there are more general magnetic fields on M, described by degenerate 2-forms, but nondegeneracy is crucial for certain isospectrality results (*c.f.* Remark 3.2). We will see in Lemma 3.1 that the spectra of the operators $\widehat{H}_k$ are independent of the choice of the connection ∇ with curvature $-2\pi i\omega$. Hence the spectra depend only on ω, h, and of course k, and will be denoted by $Spec(k\omega, h)$. (This independence of the choice of connection is special to our setting of flat tori with translation-invariant nondegenerate ω.) We will say that (M, h_1, ω_1) and (M, h_2, ω_2) are *quantum equivalent* if $Spec(k\omega_1, h_1) = Spec(k\omega_2, h_2)$ for all $k \in \mathbb{Z}^+$.

Our main results are:

Theorem 1.1. *Let ω be any translation-invariant symplectic structure on $M := \mathbb{Z}^{2m}\backslash\mathbb{R}^{2m}$. Then every translation-invariant metric h on M lies in a continuous family $\{h_t\}$ of mutually nonisometric flat metrics such that (M, h_t, ω) is classically equivalent to (M, h, ω) for all t. Moreover, if ω represents an integer cohomology class, then these (M, h_t, ω) are also quantum equivalent to (M, h, ω) for all t.*

For the remainder of the results, we assume that the forms ω_i ($i = 1, 2$) represent integer cohomology classes. In Theorem 4.6, we give necessary and sufficient conditions for quantum equivalence of pairs (M, h_1, ω_1) and (M, h_2, ω_2), and we observe that in our setting, for any choice of ω as above, $Spec(\omega, h)$ determines $Spec(k\omega, h)$ for all $k \in \mathbb{Z}^+$.

We will say that (M, h, ω) is *Kähler,* or that (h, ω) is a *Kähler structure* on M, if there exists a complex structure J such that (M, h, ω, J) is Kähler.

We then prove the following, for $M = \mathbb{Z}^{2m}\backslash\mathbb{R}^{2m}$ with m arbitrary:

Theorem 1.2. *For any translation-invariant symplectic form ω and translation-invariant metric h on M, the spectrum $Spec(\omega, h)$ determines whether (M, h, ω) is Kähler. Moreover, all translation-invariant Kähler structures (h, ω) of given volume on M are quantum equivalent. (Here both ω and h are allowed to vary.)*

Theorem 1.3. *The collection $Spec(k\omega, h)$, $k \in \mathbb{Z}^+$, does not determine the symplectic structure ω on M nor the symplectic structure $\Omega = \omega_0 + \pi^*\omega$ on T^*M (nor the restriction of Ω to the cotangent bundle with the zero section removed). In particular, quantum equivalent systems need not have the same classical phase space.*

We pause to clarify the notion of classical phase space used here and to motivate the parenthetical remark in Theorem 1.3. By considering the entire cotangent bundle, instead of the cotangent bundle minus its zero section, we are using a somewhat stronger notion of equivalence than is sometimes considered in the mathematical literature. Indeed, our notion of classical equivalence (Definition 2.1) implies that if (M_1, h_1, ω_1) and (M_2, h_2, ω_2) are classically equivalent, then (M_1, ω_1) and (M_2, ω_2) are symplectomorphic. The removal of the zero section is mathematically rather than physically motivated. Often analytical considerations necessitate replacing the Hamiltonian flow by a reparametrization that is not well behaved on the zero section. This is the case, for example, in the analysis of the singularities of the wave trace [2] and in the study of regularizations of the Kepler flow [7], [8]. Removing the zero section also results in stronger — and more difficult — geodesic rigidity results, as in the article [1] cited below. On the other hand, in classical mechanics, the phase space is the space of all possible states of the system. For a particle moving on a manifold under the influence of a magnetic field, an initial condition consisting of a given position and zero momentum (i.e., an element of the zero section of T^*M) is perfectly acceptable. While the results above were stated using the phase space (T^*M, Ω), they remain true if one removes the zero section from T^*M. In particular, the resulting stronger version of Theorem 1.3 (the parenthetical comment) is proven in Proposition 4.16.

Theorem 1.1 contrasts sharply with the case $\omega = 0$. C. Croke and B. Kleiner [1] showed that the geodesic flow on a torus is C^0 rigid, i.e., that any Riemannian manifold whose geodesic flow is C^0 conjugate to that of a flat torus (M, h) is isometric to the torus (M, h). Note that C^0-conjugacy is a much weaker condition than classical equivalence.

This is the second of two articles addressing questions of quantum equivalence. In the first [4], we constructed examples of pairs (or finite families) of Hermitian locally symmetric spaces M_i for which the line bundles with Chern class defined by the Kähler structure and their tensor powers over the various M_i are isospectral for all i.

This article was motivated by results of [3]. In fact, our results on quantum equivalence of magnetic fields are a reinterpretation and expansion of Corollaries 3.8 and 3.9 of [3].

2. Classical equivalence of magnetic flows

Definition 2.1. Given a Riemannian manifold (M, h) and a closed 2-form ω on M (which we will always assume to be nondegenerate), let Ω be the symplectic structure on T^*M given by $\Omega := \omega_0 + \pi^*\omega$, where ω_0 is the Liouville form (i.e., $\omega_0 = -d\lambda$, where λ is the canonical 1-form on T^*M) and $\pi : T^*M \to M$ is the projection. Define $H : T^*M \to \mathbb{R}$ by $H(q, \xi) = \frac{1}{2}h_q(\xi, \xi)$. We will refer to (T^*M, Ω, H) as the classical Hamiltonian system associated with (M, h, ω). Given Riemannian manifolds (M_i, h_i), $i = 1, 2$, and closed 2-forms ω_i on M_i, we will say that (M_1, h_1, ω_1) and (M_2, h_2, ω_2) are *classically equivalent* if the associated

Hamiltonian systems (T^*M_i, Ω_i, H_i) are equivalent, i.e., if there exists a symplectomorphism $\Phi : (T^*M_1, \Omega_1) \to (T^*M_2, \Omega_2)$ such that $H_1 = H_2 \circ \Phi$.

Theorem 2.2. *Let ω be a translation-invariant symplectic structure on $\mathbb{R}^{2m}$, let A be a linear symplectomorphism of $(\mathbb{R}^{2m}, \omega)$, let h be a translation-invariant metric on $\mathbb{R}^{2m}$, and let $\mathcal{L}$ be a lattice in $\mathbb{R}^{2m}$. We will continue to denote by ω and h the induced structures on quotients of $\mathbb{R}^{2m}$ by a lattice. Then $(\mathcal{L}\backslash\mathbb{R}^{2m}, h, \omega)$ is classically equivalent to $(A(\mathcal{L})\backslash\mathbb{R}^{2m}, h, \omega)$.*

Remarks 2.3.
(i) The conclusion may be rephrased as the statement that $(\mathcal{L}\backslash\mathbb{R}^{2m}, A^*h, \omega)$ is classically equivalent to $(\mathcal{L}\backslash\mathbb{R}^{2m}, h, \omega)$.
(ii) In Theorem 2.2, we do not require that $\mathcal{L}$ have maximal rank in $\mathbb{R}^{2m}$, i.e., that $\mathcal{L}\backslash\mathbb{R}^{2m}$ be a torus. However, in the case that it is a torus and that ω represents an integer cohomology class in $\mathcal{L}\backslash\mathbb{R}^{2m}$, the reformulation in (i) will give us different quantum Hamiltonians (Laplacians associated with different metrics) on the same complex line bundle. We will see in Corollary 4.8 that the systems are quantum equivalent.

Proof. Let $n = 2m$. Under the standard identification of $T^*\mathbb{R}^n$ with $\mathbb{R}^{2n}$, the symplectic form $\Omega = \omega_0 + \pi^*\omega$ is a translation-invariant 2-form and thus may be identified with the bilinear form on $\mathbb{R}^{2n}$ with matrix

$$\begin{bmatrix} C & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{bmatrix}$$

with respect to the standard basis, where each block is of size $n \times n$ and where C is the matrix of the anti-symmetric nondegenerate bilinear form on $\mathbb{R}^n$ defined by ω. The linear map $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ given by

$$\Phi(q, p) = (Aq + C^{-1}({}^tA^{-1} - \mathrm{Id})p, p)$$

preserves Ω, as can be seen by an easy computation using ${}^tC = -C$ and ${}^tACA = C$. The Hamiltonian H depends only on p (since h is translation invariant) and thus is also preserved by Φ. Thus Φ is a self-equivalence of the Hamiltonian system $(T^*\mathbb{R}^n, h, \omega)$. Finally, we have $\Phi(q_0 + q, p) = (Aq_0, 0) + \Phi(q, p)$ for all $q_0 \in \mathbb{R}^n$ and, in particular, for all $q_0 \in \mathcal{L}$. Thus Φ induces an equivalence between $(\mathcal{L}\backslash\mathbb{R}^{2m}, h, \omega)$ and $(A(\mathcal{L})\backslash\mathbb{R}^{2m}, h, \omega)$. $\square$

Corollary 2.4. *Let ω be a translation-invariant symplectic structure on a torus $M = \mathbb{Z}^{2m}\backslash\mathbb{R}^{2m}$. Then every translation-invariant Riemannian metric h on M belongs to a continuous family $\{h_t\}_t$ of mutually nonisometric translation-invariant Riemannian metrics such that (M, h_t, ω) is classically equivalent to (M, h, ω) for all t. The parameter space of this deformation has dimension at least $2m$.*

Proof. h_t is defined as A_t^*h, where A_t (with $A_0 = \mathrm{Id}$) is a curve in the group of linear isomorphisms of $\mathbb{R}^{2m}$ that preserve ω. This group is isomorphic to *Sp*$(2m, \mathbb{R})$ and has dimension $m(2m + 1)$, while the group of linear isomorphisms that preserve h is isomorphic to $O(2m)$ and has dimension $m(2m - 1)$. The corollary thus follows from Remark 2.3(i). $\square$

3. Hermitian line bundles over tori

3.1. **Hermitian connections with the same translation-invariant curvature.** Let M be a compact C^∞ manifold, L a Hermitian line bundle over M, and let $[\omega] \in H^2(M;\mathbb{Z})$ be the Chern class of L. The curvature of any Hermitian connection on L lies in $-2\pi i[\omega]$. (Our notation differs from that of [4] by a factor of 2π.) If ∇ and ∇' are two Hermitian connections on L, then $\nabla' = \nabla + 2\pi i\beta$ for some real-valued 1-form β on M. The two connections have the same curvature if and only if $d\beta = 0$, in which case $\beta = \alpha + df$ for some harmonic 1-form α and some $f \in C^\infty(M)$. The term df changes the connection only by a gauge equivalence: in fact, letting $\mathcal{E}(L)$ denote the space of smooth sections of L, then the map $\mathcal{E}(L) \to \mathcal{E}(L)$ given by $s \mapsto e^{2\pi i f}s$ intertwines $\nabla + 2\pi i df$ and ∇. Given any Riemannian metric h on M, this map also intertwines the Laplacians $-\,\textit{trace}(\nabla + 2\pi i df)^2$ and $-\,\textit{trace}(\nabla^2)$. Thus the two Laplacians are isospectral. The same statement holds for the associated Laplacians on all the higher tensor powers of L. Thus we may assume that $f = 0$.

In general, the addition of a harmonic 1-form $2\pi i\alpha$ to ∇ will affect the spectrum. However, we will see that in the case of line bundles with nondegenerate Chern class over flat tori, endowed with a connection whose curvature form on the torus is translation invariant, the addition of a harmonic term does *not* affect the spectrum; see Lemma 3.1 below. Thus in this case, the spectrum of the Laplacian depends only on the metric on the torus and the curvature of the connection on the bundle.

3.2. **Principal circle bundles over tori.** Let $M = \mathbb{Z}^{2m}\backslash\mathbb{R}^{2m}$, where m is a positive integer. Let ω be a translation-invariant symplectic structure on M that represents an integer cohomology class. We will first construct a principal circle bundle P with Chern class $[\omega]$. The bundle P will be a quotient by a discrete subgroup of a two-step nilpotent Lie group N, isomorphic to the Heisenberg group of dimension $2m+1$.

Since ω is translation invariant, it may be viewed as a nondegenerate antisymmetric bilinear map $\omega : \mathbb{R}^{2m} \times \mathbb{R}^{2m} \to \mathbb{R}$ that takes integer values on $\mathbb{Z}^{2m} \times \mathbb{Z}^{2m}$. We endow $N := \mathbb{R}^{2m+1}$ with the structure of a 2-step nilpotent Lie group with multiplication

$$(u_1, t_1)(u_2, t_2) = (u_1 + u_2, t_1 + t_2 + \tfrac{1}{2}\omega(u_1, u_2))$$

for all $u_1, u_2 \in \mathbb{R}^{2m}$ and $t_1, t_2 \in \mathbb{R}$. Then N is isomorphic to the $(2m+1)$-dimensional Heisenberg group. The coordinate vector field $Z := \frac{\partial}{\partial t}$ is left invariant and spans the center $\mathfrak{z} = \{0\} \times \mathbb{R}$ of the Lie algebra $\mathfrak{n}$ of N. The center coincides with the derived algebra, so the Lie bracket may be viewed as a bilinear map $[\ ,\ ] : \mathbb{R}^{2m} \times \mathbb{R}^{2m} \to \mathfrak{z}$, which is given by

$$[X, Y] = \omega(X, Y)Z.$$

Let $\Gamma \subset N$ be the subgroup generated by $(e_1, 0), \ldots, (e_{2m}, 0), (0, 1) \in \mathbb{R}^{2m+1} = N$, where e_j denotes the jth standard basis vector of $\mathbb{R}^{2m}$. Then the projection of Γ to $\mathbb{R}^{2m}$ is $\mathbb{Z}^{2m}$. The intersection of Γ with the center $\{0\} \times \mathbb{R}$ of N is precisely $\{0\} \times \mathbb{Z}$, the subgroup generated by the element $(0, 1)$. In fact, for $X, Y \in \{\pm e_1, \ldots, \pm e_{2m}\}$ the commutator $(X, 0)(Y, 0)(X, 0)^{-1}(Y, 0)^{-1}$ equals $(X + Y, \frac{1}{2}\omega(X, Y))(-X - Y, \frac{1}{2}\omega(-X, -Y)) = (0, \omega(X, Y))$ which lies in $\{0\} \times \mathbb{Z}$ since ω is integer valued on $\mathbb{Z}^{2m} \times \mathbb{Z}^{2m}$; moreover, any product $(X_1, 0) \cdot \ldots \cdot (X_k, 0)$ with $X_1, \ldots, X_k \in \{\pm e_1, \ldots, \pm e_{2m}\}$ and $X_1 + \ldots + X_k = 0$ can be written as a product of commutators as above.

In particular, Γ is a uniform discrete subgroup of N. Set $P = \Gamma \backslash N$. The center of N projects to a circle, and the action of the center by translations on N induces a circle action on P, giving P the structure of a principal circle bundle over M.

We identify the circle S^1, given by the quotient of the center of N by its intersection with Γ, with the unitary group $U(1)$. Its Lie algebra is thus identified with the space of purely imaginary complex numbers. Under this identification, the vector $Z \in \mathfrak{z}$ above corresponds to $2\pi i \in i\mathbb{R} = T_1U(1)$; hence, a connection on P is specified by an S^1-invariant 1-form $2\pi i\mu$ on P such that $2\pi i\mu(Z) \equiv 2\pi i$; that is, $\mu(Z) \equiv 1$. (Here μ is real-valued.) The kernel $\mathcal{H}$ of μ is called the horizontal distribution associated with the connection. By abuse of terminology, we will say that μ is *left invariant* if it pulls back to a left-invariant 1-form on N. In this case, $\mathcal{H}$ is spanned by left-invariant vector fields (again in the sense that a left-invariant vector field on N induces a well-defined vector field on $P = \Gamma \backslash N$, which we refer to as left invariant) and thus may be viewed as a subspace of $\mathfrak{n}$ complementary to $\mathfrak{z}$. Conversely, since every left-invariant 1-form is also S^1 invariant, any complement of $\mathfrak{z}$ in $\mathfrak{n}$ is the horizontal distribution associated with some translation-invariant connection on P.

Suppose that $2\pi i\mu$ is a left-invariant connection on P. For $X, Y \in \mathcal{H}$, we have

$$2\pi i d\mu(X, Y) = -2\pi i\mu([X, Y]) = -2\pi i\mu(\omega(X, Y)Z) = -2\pi i\omega(X, Y)$$

since $\mu(Z) = 1$. Thus every translation-invariant connection on P has curvature form $-2\pi i\omega$.

Let $\alpha : \mathbb{R}^{2m} \to \mathbb{R}$ be a linear functional. Because of the nondegeneracy of ω, the map $\mathfrak{n} \to \mathfrak{n}$ that sends $X \in \mathbb{R}^{2m}$ to $\alpha(X)Z$ and sends Z to zero is an inner derivation of $\mathfrak{n}$, and the map $N \to N$ given by $(u, t) \mapsto (u, t + \alpha(u))$ is an inner automorphism of N.

It follows that if μ' is another left-invariant 1-form such that $\mu'(Z) = 1$, then $\mu' = \mu \circ \mathrm{Ad}(a)$ for some $a \in N$, and the corresponding horizontal distribution satisfies $\mathcal{H}' = \mathrm{Ad}(a^{-1})\mathcal{H}$.

3.3. **Associated Hermitian line bundles.** Let ω and P be as above and let $2\pi i\mu$ be a left-invariant connection on P. The group $S^1 = U(1)$ acts on $\mathbb{C}$ in the standard way, hence diagonally on the product $P \times \mathbb{C}$, giving rise to a Hermitian line bundle

$$L = (P \times \mathbb{C})/\sim$$

where $\sim$ is the equivalence relation given by $(p, w) \sim (pz^{-1}, zw)$ for $p \in P$, $w \in \mathbb{C}$, and $z \in S^1 = U(1)$. The bundle L has Chern class $[\omega]$.

The higher tensor powers of L are given by

$$L^{\otimes k} = (P \times \mathbb{C})/\sim_k$$

where $\sim_k$ is given by $(p, w) \sim_k (pz^{-1}, z^k w)$ for $p \in P$, $w \in \mathbb{C}$, and $z \in S^1 = U(1)$.

The space $C^\infty(M, L^{\otimes k})$ of smooth sections of $L^{\otimes k}$ may be identified with the subspace $C_k^\infty(P, \mathbb{C})$ given by

$$C_k^\infty(P, \mathbb{C}) = \{f \in C^\infty(P, \mathbb{C}) \mid f(pz^{-1}) = z^k f(p) \text{ for all } p \in P, z \in S^1 = U(1)\}. \tag{3.1}$$

Equivalently,

$$C_k^\infty(P,\mathbb{C}) = \{f \in C^\infty(P,\mathbb{C}) \mid Zf = -2\pi i k f\}. \tag{3.2}$$

Because of the trivialization $TP \cong P \times \mathfrak{n}$, any complex 1-form on P may be viewed as a map from $\mathfrak{n}$ to the space of smooth complex functions on P. For $f \in C_k^\infty(P,\mathbb{C})$, the map corresponding to the 1-form $df + 2\pi i k f \mu$ actually maps $\mathfrak{n}$ to $C_k^\infty(P,\mathbb{C})$ and vanishes on $\mathfrak{z}$; hence, it induces a well-defined map from $\mathbb{R}^{2m}$ to $C_k^\infty(P,\mathbb{C})$. Recalling Equation (3.1) and identifying $\mathbb{R}^{2m}$ with the tangent space at each point of M, we thus get a map $\nabla f : TM \to C^\infty(M, L^{\otimes k})$. This defines the Hermitian connection ∇ on $L^{\otimes k}$ associated with the connection $2\pi i\mu$ on the principal bundle. (Here we are using the same notation ∇ for the connection on each of the bundles $L^{\otimes k}$. The connection ∇ on $L^{\otimes k}$ is of course the usual connection on the kth tensor power of the bundle L arising from the connection ∇ on L.) For $X \in TM$ and $\tilde{X}$ any horizontal vector in TP with $\pi_*\tilde{X} = X$, where $\pi : P \to M$ is the bundle projection, we have

$$\nabla_X f = \tilde{X} f.$$

The curvature of ∇ is $-2\pi i k\omega$.

Given a flat metric h on M (i.e., an inner product on $\mathbb{R}^{2m}$), let $\{X_1, \dots, X_{2m}\}$ be an orthonormal basis of the Lie algebra $\mathbb{R}^{2m}$ of M, and let $\tilde{X}_1, \dots, \tilde{X}_{2m}$ be the horizontal lifts to vector fields on the principal bundle P. Then under the identification of $C^\infty(M, L^{\otimes k})$ with $C_k^\infty(P,\mathbb{C})$ as in Equation (3.1), the Laplacian on $C^\infty(M, L^{\otimes k})$ defined by the connection ∇ is given by

$$\Delta(f) = -\sum_{j=1}^{2m} \tilde{X}_j^2(f).$$

Let ρ denote the representation of the nilpotent Lie group N on $L^2(P)$ given by $(\rho(a)f)(p) = f(pa)$ and let ρ_* be the representation of the Lie algebra $\mathfrak{n}$ given by the differential of ρ. Then by Fourier decomposition with respect to the action of the center of N, we have

$$L^2(P) = \oplus_{k\in\mathbb{Z}} L_k^2(P)$$

where

$$L_k^2(P) = \{f \in L^2(P) \mid \rho(z^{-1})f = z^k f \text{ for all } z \in S^1 = U(1)\}.$$

I.e., $L_k^2(P)$ is the closure of $C_k^\infty(P,\mathbb{C})$ in $L^2(P)$. Given a translation-invariant connection on L (and thus on $L^{\otimes k}$ for all $k \in \mathbb{Z}^+$) and a flat metric on M, the associated Laplacian, viewed as an operator on $C_k^\infty(P,\mathbb{C})$, extends to $L_k^2(P)$ as the densely defined operator

$$\Delta = -\sum_{j=1}^{2m} \rho_*(\tilde{X}_j)^2. \tag{3.3}$$

Lemma 3.1. *We continue to assume that ω is a translation-invariant symplectic structure on the torus M and that the cohomology class of ω is integral. Let L be a Hermitian line bundle with Chern class $[\omega]$, and let ∇ and ∇' be two connections on L with curvature $-2\pi i\omega$. Then* *given any flat metric on M, the Laplacians, and thus the quantum Hamiltonians, on $L^{\otimes k}$ defined by ∇ and ∇' are isospectral for all $k \in \mathbb{Z}^+$.*

Proof. Note that L is determined by its Chern class up to a bundle isomorphism inducing the identity map on M. Such isomorphisms preserve the curvature forms of the connections which they intertwine, and the spectra of the corresponding bundle Laplacians coincide. Therefore, we may assume that L is the Hermitian line bundle which we explicitly constructed above.

Let ∇ be the connection associated with the principal connection $2\pi i\mu$ as above. By the discussion in Subsection 3.1, we may assume that $\nabla' = \nabla + 2\pi i\alpha$ for some harmonic 1-form α on M. Viewing α as a linear functional on $\mathbb{R}^{2m}$, the map $\mathfrak{n} \to \mathfrak{n}$ given by $X + cZ \mapsto \alpha(X)Z$ (for all $X \in \mathbb{R}^{2m}$ and $c \in \mathbb{R}$) is an inner derivation and ∇' is the connection on L associated with a principal connection $2\pi i\mu \circ \mathrm{Ad}(a)$ for some $a \in N$. The horizontal distribution $\mathcal{H}'$ is given by $\mathrm{Ad}(a^{-1})\mathcal{H}$. It follows that the Laplacian associated with ∇' on $C_k^\infty(P, \mathbb{C})$ is given by

$$\Delta' = \sum_{j=1}^{2m} \rho_*(\mathrm{Ad}(a^{-1})\tilde{X}_j)^2 = \sum_{j=1}^{2m} \rho(a^{-1})\rho_*(\tilde{X}_j)^2\rho(a) = \rho(a^{-1}) \circ \Delta \circ \rho(a).$$

□

Remark 3.2. The hypothesis of nondegeneracy of ω is essential here. At the other extreme in which $\omega = 0$ so that L is the trivial bundle, the spectra of the various Laplacians $-(d - 2\pi i\alpha)^2$ associated with the (harmonic) connections of curvature zero form the Bloch spectrum of the torus.

Notation 3.3. In the notation of Lemma 3.1, we will write

$$\textit{Spec}(k\omega, h)$$

for the spectrum of the operator $\widehat{H}_k = \frac{\hbar^2}{2}\Delta$, where $\hbar = \frac{1}{k}$ and Δ is the Laplacian on $L^{\otimes k}$ defined by the flat metric h on $\mathbb{Z}^{2m}\backslash\mathbb{R}^{2m}$ and any connection ∇ on L with curvature $-2\pi i\omega$. By the lemma, this spectrum is well defined.

4. Quantum equivalent line bundles

Notation 4.1. Denote the standard coordinates on $\mathbb{R}^{2m}$ by $(x, y) = (x_1, \dots, x_m, y_1, \dots, y_m)$. Given an m-tuple $\mathbf{r} = (r_1, \dots, r_m)$ of positive integers such that

$$r_1 \mid r_2 \mid \dots \mid r_m, \tag{4.1}$$

define a translation-invariant symplectic form $\omega_{\mathbf{r}}$ on $\mathbb{R}^{2m}$ by

$$\omega_{\mathbf{r}} = \sum_{j=1}^{m} r_j \, dx_j \wedge dy_j.$$

Proposition 4.2. [6, p. 304] *Let ω be a translation-invariant symplectic structure on $\mathbb{R}^{2m}$ such that $[\omega] \in H^2(M; \mathbb{Z})$. Then there exists a unique m-tuple $\mathbf{r}$ satisfying Equation 4.1 such that $A^*\omega = \omega_{\mathbf{r}}$ for some $A \in SL(2m, \mathbb{Z})$. We refer to the entries of this m-tuple as the* Chern invariant factors.

Thus by a linear change of coordinates preserving $\mathbb{Z}^{2m}$, we may assume when convenient that $\omega = \omega_{\mathbf{r}}$ for some $\mathbf{r}$ satisfying 4.1.

Remark 4.3. Line bundles are of course classified by their Chern classes, not by the Chern invariant factors. The m-tuple $\mathbf{r}$ is a complete homeomorphism invariant of ω, in the sense that given two integral symplectic structures with the same Chern invariant factors, there is a self-homeomorphism of the base space pulling back one integral symplectic structure to the other; however, integral symplectic structures with the same Chern invariant factors need not be cohomologous and thus may give rise to inequivalent line bundles.

Notation 4.4.

(i) Given a translation-invariant symplectic structure ω and a translation-invariant Riemannian metric h on $\mathbb{R}^{2m}$, viewed as bilinear forms, define a linear transformation $F : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ by the condition

$$\omega(u, v) = h(F(u), v)$$

for all $u, v \in \mathbb{R}^{2m}$. Let $\boldsymbol{h}$ and $\boldsymbol{\omega}$ denote the Gram matrices of the bilinear forms h and ω with respect to the standard basis of $\mathbb{R}^{2m}$. The matrix of the linear transformation F in this basis is given by

$$\boldsymbol{F} = \boldsymbol{h}^{-1}\boldsymbol{\omega}.$$

Note that F is antisymmetric relative to the inner product h, and its eigenvalues are purely imaginary; we denote them by $\pm d_1^2 i, \dots, \pm d_m^2 i$.

The linear transformation F may be expressed in terms of the "musical isomorphisms": Given a finite-dimensional real vector space V and a nondegenerate bilinear form $B : V \times V \to \mathbb{R}$, denote by $B^\flat : V \to V^*$ the isomorphism from V to its dual space given by $B^\flat(u) = B(\cdot, u)$, i.e., $(B^\flat(u))(v) = B(v, u)$ for $u, v \in V$, and by $B^\sharp : V^* \to V$ the inverse of $B^\flat$. Then $F = h^\sharp \circ \omega^\flat$.

(ii) Let $M = \mathbb{Z}^{2m}\backslash\mathbb{R}^{2m}$. Set

$$V_\omega = \sqrt{\det(\boldsymbol{\omega})} \; = \int_M \frac{1}{m!}\omega^m \,,$$

the *symplectic volume* of M. Since the standard basis of $\mathbb{R}^{2m}$ is a basis of $\mathbb{Z}^{2m}$ we have, in particular, $V_{\omega_{\mathbf{r}}} = r_1 r_2 \dots r_m$.

Proposition 4.5. *We use Notation 3.3 and 4.4. Let $M = \mathbb{Z}^{2m}\backslash\mathbb{R}^{2m}$, let ω be a translation-invariant symplectic structure on M representing an integer cohomology class, and let h be any flat metric on M. Given an m-tuple $\mathbf{j} = (j_1, \dots, j_m)$ of nonnegative integers, let*

$$\nu(\mathbf{j}) = \pi \sum_{i=1}^{m} d_i^2(2j_i + 1).$$

Then Spec$(k\omega, h)$ is the collection of all $\frac{1}{k}\nu(\mathbf{j})$, $\mathbf{j} \in \mathbb{N}_0^m$, each counted $k^m V_\omega$ times.

Proof. Recalling Notation 3.3, we see that $2k^2\,\mathit{Spec}(k\omega, h)$ is the spectrum of the operator in Equation (3.3) acting on $L^2_k(P)$. Rather than carry out the computation here, we refer to [5], Section 3, where a similar computation is performed. We indicate here how to translate the computation in [5] to our setting. We assume that $\omega = \omega_{\mathbf{r}}$ for some $\mathbf{r} = (r_1, \dots, r_m)$ as above. Perform a change of coordinates on $\mathbb{R}^{2m}$, letting $x_i' = r_i x_i$ and $y_i' = y_i$ for $i = 1, \dots, m$. In these new coordinates, $\omega = \sum_{i=1}^m dx_i' \wedge dy_i'$, and the lattice $\mathbb{Z}^{2m}$ is the collection of all elements with coordinates in $r_1\mathbb{Z} \times \cdots \times r_m\mathbb{Z} \times \mathbb{Z}^m$. This change of coordinates aligns our notation with that in [5]. Next, in [5], the operator under study is the Laplacian Δ_P associated with the Riemannian metric on the Heisenberg manifold $P = \Gamma\backslash N$ induced by the left-invariant metric on N for which the basis $\{\tilde{X}_1, \dots, \tilde{X}_{2m}, Z\}$ is orthonormal, where $\tilde{X}_1, \dots, \tilde{X}_{2m}$ are as in Subsection 3.3. We have $\Delta_P = \Delta + (\rho_* Z)^2$ for Δ as in Equation (3.3). (In the notation of [5], we are setting g_{2m+1} equal to 1.) Writing $L^2(P) = \oplus_{k\in\mathbb{Z}} L^2_k(P)$, then it is shown in [5] that for $k \neq 0$, the spectrum of Δ_P restricted to $L^2_k(P)$ is given by the collection of numbers $4\pi^2k^2 + 2|k|\nu(\mathbf{j})$, each occurring with multiplicity $|k|^m r_1 \dots r_m$. (Our $|k|$ is denoted by c in [5].) The operator $(\rho_* Z)^2$ acts on $L^2_k(P)$ as multiplication by $4\pi^2k^2$. Correcting for this term and taking $k \in \mathbb{Z}^+$, we obtain the proposition. □

Theorem 4.6. *We use Notation 3.3 and 4.4. Let ω and ω' be two translation-invariant symplectic structures on M representing integer cohomology classes, and let h and h' be flat metrics on M. Then the following are equivalent:*

(i) *$\mathit{Spec}(\omega, h) = \mathit{Spec}(\omega', h')$.*
(ii) *$\mathit{Spec}(k\omega, h) = \mathit{Spec}(k\omega', h')$ for all $k \in \mathbb{Z}^+$.*
(iii) *The linear transformations $h^\sharp \circ \omega^\flat$ and $h'^\sharp \circ \omega'^\flat$ (equivalently the matrices $\boldsymbol{h}^{-1}\boldsymbol{\omega}$ and $\boldsymbol{h'}^{-1}\boldsymbol{\omega'}$) have the same eigenvalue spectrum, and $V_\omega = V_{\omega'}$.*

Proof. It is clear from Proposition 4.2 that (i) and (ii) are equivalent and that (iii) implies (i) and (ii). To see that (i) implies (iii), note that the lowest eigenvalue occurring in $\mathit{Spec}(\omega, h)$ is $\pi(d_1^2 + \cdots + d_m^2)$, with multiplicity precisely V_ω. Thus V_ω is spectrally determined. If we order the d_j so that $d_1^2 \le d_2^2 + \cdots \le d_m^2$, then $2\pi d_1^2$ is the difference between the first two distinct eigenvalues μ_1 and μ_2. From the multiplicity of μ_2, we can determine how many of the d_j^2 equal d_1^2; denote this number by p. Since we know $\sum_{j=1}^m d_j^2$ from μ_1 and we know d_1^2, we can determine all eigenvalues $\nu(\mathbf{j})$ for which $j_{p+1} = \cdots = j_m = 0$, along with their multiplicities. Removing all these from the spectrum, the lowest remaining eigenvalue is $\nu(\mathbf{j})$ where $j_{p+1} = 1$ and all other j_l's are zero. This enables us to determine d_{p+1}^2 and its multiplicity, and we continue inductively. □

Remark 4.7. If the symplectic volumes V_ω and $V_{\omega'}$ coincide, then the first part of condition (iii) in the previous theorem can be replaced by the condition that $\boldsymbol{h}$ and $\boldsymbol{h'}$ have the same determinant, or equivalently that $\mathit{vol}(M, h) = \mathit{vol}(M, h')$, since the determinant is multiplicative and $V_\omega = \sqrt{\det \boldsymbol{\omega}}$.

Corollary 4.8. *Let ω be a translation-invariant symplectic structure on $M = \mathbb{Z}^{2m}\backslash\mathbb{R}^{2m}$ that represents an integer cohomology class. Given any translation-invariant metric h on M, let*

$\{h_t\}_t$ be a family of metrics constructed as in the proof of Corollary 2.4. Then (M, h_t, ω) is quantum equivalent as well as classically equivalent to (M, h, ω) for all t.

Proof. Classical equivalence was shown in Corollary 2.4. Quantum equivalence follows from Theorem 4.6; in fact, if A_t (and hence A_t^{-1}) preserves ω and $h_t = A_t^* h$ then we have

$$\boldsymbol{h}_t^{-1}\boldsymbol{\omega} = \boldsymbol{A}_t^{-1}\boldsymbol{h}^{-1}\boldsymbol{\omega}\boldsymbol{A}_t.$$

$\square$

Definition 4.9. We will say that (M, h, ω) is *Kähler*, or that the pair (h, ω) is a *Kähler structure* on M, if there exists a complex structure J on M such that (M, h, J) is a Kähler manifold whose associated Kähler form is ω.

Proposition 4.10. *The tuple (M, h, ω) is Kähler if and only if all the eigenvalues of $h^\sharp \circ \omega^\flat$ (equivalently, of the matrix $\boldsymbol{h}^{-1}\boldsymbol{\omega}$) are $\pm i$.*

Proof. The latter condition is equivalent to $F^2 = -\operatorname{Id}$ for the h-antisymmetric map $F = h^\sharp \circ \omega^\flat$ from Notation 4.4(i). But this is equivalent to (M, h, ω) being Kähler (with complex structure F). $\square$

Corollary 4.11. *Spec(ω, h) determines whether (M, h, ω) is Kähler. Moreover, any two Kähler structures (h, ω) and (h', ω') that have the same volume are quantum equivalent.*

Note that in the Kähler case, the symplectic volume V_ω equals the Riemannian volume of (M, h); recall Remark 4.7 together with Proposition 4.10.

For the construction of examples, we will restrict attention to metrics of the form

$$h_{\mathbf{a},\mathbf{b}} = \sum_{j=1}^{m} (a_j^2\, dx_j^2 + b_j^2\, dy_j^2),$$

when $\omega = \omega_{\mathbf{r}} = \sum_{j=1}^m r_j\, dx_j \wedge dy_j$, where $\mathbf{a} = (a_1, \dots, a_m)$, $\mathbf{b} = (b_1, \dots, b_m) \in \mathbb{R}^m$.

Remark 4.12. The eigenvalues of $\boldsymbol{h}_{\mathbf{a},\mathbf{b}}^{-1}\,\boldsymbol{\omega}_{\mathbf{r}}$ are given by $\pm i\frac{r_1}{a_1 b_1}, \dots, \pm i\frac{r_m}{a_m b_m}$. The symplectic volume $V_{\omega_{\mathbf{r}}}$ equals $r_1 r_2 \dots r_m$, by Notation 4.4(ii).

Examples 4.13.
(i) Let $m = 2$. Set $h = dx_1^2 + dy_1^2 + dx_2^2 + 4\, dy_2^2$, so the representing matrix is

$$\boldsymbol{h} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix},$$

and let

$$\omega = 2\, dx_1 \wedge dy_1 + 2\, dx_2 \wedge dy_2 \quad \text{and} \quad \omega' = dx_1 \wedge dy_1 + 4\, dx_2 \wedge dy_2.$$

Then both $\boldsymbol{h}^{-1}\boldsymbol{\omega}$ and $\boldsymbol{h}^{-1}\boldsymbol{\omega'}$ have eigenvalues $\pm i$ and $\pm 2i$. Thus ω and ω' are quantum equivalent magnetic fields on $(\mathbb{Z}^4\backslash\mathbb{R}^4, h)$. The two structures $(\mathbb{Z}^4\backslash\mathbb{R}^4, h, \omega)$ and $(\mathbb{Z}^4\backslash\mathbb{R}^4, h, \omega')$ are not Kähler.

(ii) Set $h = dx_1^2 + 4\,dy_1^2 + dx_2^2 + 4\,dy_2^2$, so the representing matrix is

$$\boldsymbol{h} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix},$$

$$\omega = 2\,dx_1 \wedge dy_1 + 2\,dx_2 \wedge dy_2,$$

$h' = dx_1^2 + dy_1^2 + 4\,dx_2^2 + 4\,dy_2^2$, so the representing matrix is

$$\boldsymbol{h'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix},$$

and

$$\omega' = dx_1 \wedge dy_1 + 4\,dx_2 \wedge dy_2.$$

Then all eigenvalues of $\boldsymbol{h}^{-1}\boldsymbol{\omega}$ and of $\boldsymbol{h'}^{-1}\boldsymbol{\omega'}$ are $\pm i$. Thus $(\mathbb{Z}^4\backslash\mathbb{R}^4, h, \omega)$ and $(\mathbb{Z}^4\backslash\mathbb{R}^4, h', \omega')$ are quantum equivalent Kähler structures. Note that h and h' are isometric via the map that interchanges the coordinates y_1 and x_2; so ω and the corresponding pullback of ω' can be viewed as quantum equivalent Kähler structures on the same underlying Riemannian manifold.

Remark 4.14. The first of the two examples above first appeared in a slightly different context in [3].

In examples of pairs of quantum equivalent line bundles arising from Theorem 4.6, the cotangent bundles endowed with the associated symplectic forms will in general be nonsymplectomorphic. In particular, this is the case for the pairs in Example 4.13. In fact, we have:

Proposition 4.15. *Let ω, ω' be two translation-invariant symplectic structures on the torus $M = \mathbb{Z}^{2m}\backslash\mathbb{R}^{2m}$ with Chern invariant factors $\mathbf{r} = (r_1, \ldots, r_m)$ and $\mathbf{r}' = (r_1', \ldots, r_m')$, respectively. Let $\Omega := \pi^*\omega + \omega_0$ and $\Omega := \pi^*\omega' + \omega_0$ be the associated symplectic forms on T^*M, where ω_0 is the Liouville form. If $\mathbf{r} \neq \mathbf{r}'$, then (T^*M, Ω) and (T^*M, Ω') are not symplectomorphic.*

Proof. For $k = 1, \ldots, m$, we consider the values of the integer cohomology classes of T^*M represented by $\Omega^k := \Omega \wedge \ldots \wedge \Omega$ on integer homology classes of T^*M. We have $T^*M \cong M \times \mathbb{R}^{2m}$. In particular, each integer homology class in $H_{2k}(T^*M; \mathbb{Z})$ can be represented by a suitable smooth closed cycle in $M \times \{0\}$ (a finite sum of oriented $2k$-dimensional subtori). We consider the integrals of $\Omega^k := \Omega \wedge \ldots \wedge \Omega$ over such $2k$-cycles. These are equal to the integrals of ω^k over the corresponding cycles in M. Obviously, the minimal nonzero absolute value of these integrals is $r_1 \cdot \ldots \cdot r_k$. Thus, if there were a symplectomorphism $(T^*M, \Omega) \to (T^*M, \Omega')$, then $r_1 \cdot \ldots \cdot r_k = r_1' \cdot \ldots \cdot r_k'$ for each $k = 1, \ldots, m$, and thus $\mathbf{r} = \mathbf{r}'$. $\square$

The next proposition shows that the previous result continues to hold if we remove the zero section from the cotangent bundle, as it might seem natural to do in some contexts (see the comments after Theorem 1.3): Let $T^*M \smallsetminus 0 \cong M \times (\mathbb{R}^{2m} \smallsetminus \{0\})$ denote the manifold of all nonvanishing cotangent vectors to M (this is an open submanifold of T^*M).

Proposition 4.16. *In the situation of Proposition 4.15,* $\mathbf{r} \neq \mathbf{r}'$ *also implies that* $(T^*M \smallsetminus 0, \Omega)$ *and* $(T^*M \smallsetminus 0, \Omega')$ *are not symplectomorphic.*

Proof. Let $X \in \mathbb{R}^{2m} \smallsetminus \{0\}$ be arbitrary. For $j \leq 2m-2$, the jth homology group of $T^*M \smallsetminus 0 \cong M \times (\mathbb{R}^{2m} \smallsetminus \{0\})$ is still isomorphic to the jth homology group of M, and each of its cycles can be represented by a suitable cycle in $M \times \{X\}$. Therefore, by the same argument as in the proof of Proposition 4.15 we see that the symplectomorphism class of Ω determines $r_1, \dots, r_{m-1}$. In order to see that it also determines r_m, note that $H_{2m}(M \times (\mathbb{R}^{2m} \smallsetminus \{0\}); \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}^{2m}$, where $\mathbb{Z}$ corresponds to $H_{2m}(M;\mathbb{Z})$ and $\mathbb{Z}^{2m}$ is generated by products of 1-cycles in M with a $(2m-1)$-cycle in $\mathbb{R}^{2m} \smallsetminus \{0\}$ generating $H_{2m-1}(\mathbb{R}^{2m} \smallsetminus \{0\}; \mathbb{Z})$. Since the integral of Ω^m over such products vanishes, we still have that the minimal nonzero absolute value of the integrals of Ω^m over $2m$-cycles representing integral homology classes in $T^*M \smallsetminus 0$ is $r_1 \cdot \ldots \cdot r_m$. □

References

[1] C. Croke and B. Kleiner. Conjugacy and rigidity for manifolds with a parallel vector field. *J. Differential Geom.* **39** (1994), no. 3, 659–680.

[2] J. Duistermaat and V. Guillemin. The spectrum of positive elliptic operators and periodic bicharacteristics. *Invent. Math* **29** (1975), 39–79.

[3] C. Gordon, P. Guerini, T. Kappeler, and D. Webb. Inverse spectral results on even dimensional tori. *Ann. Inst. Fourier* **58** (2008), 2245–2501.

[4] C. Gordon, W. Kirwin, D. Schueth, and D. Webb. Quantum equivalent magnetic fields that are not classically equivalent. *Ann. Inst. Fourier* **60** (2010), no. 7, 2403–2419.

[5] C. Gordon and E. Wilson. The spectrum of the Laplacian on Riemannian Heisenberg manifolds. *Michigan Math. J.* **33** (1986), no. 2, 253–271.

[6] P. Griffiths and J. Harris. *Principles of Algebraic Geometry*. John Wiley & Sons, 1978.

[7] G. Heckman and T. de Laat. On the regularization of the Kepler problem. *J. Symplectic Geom.* (to appear), `arXiv:1007.3695`

[8] T. Ligon and M. Schaaf: On the global symmetry of the classical Kepler problem, *Reports on Math. Phys.* **9** (1976), 281–300.

[9] N. M. J. Woodhouse. *Geometric Quantization*, second Edition. Oxford University Press, Inc., New York, 1991.

DEPARTMENT OF MATHEMATICS, DARTMOUTH COLLEGE, HANOVER, NH 03755
E-mail address: csgordon@dartmouth.edu

CAMGSD, DEPARTAMENTO DE MATEMÁTICA, INSTITUTO SUPERIOR TÉCNICO, AV. ROVISCO PAIS, 1049-001 LISBOA, PORTUGAL
*CURRENT ADDRESS: MATHEMATISCHES INSTITUT, UNIVERSITY OF COLOGNE, WEYERTAL 86 - 90, 50931 COLOGNE, GERMANY
E-mail address: will.kirwin@gmail.com

INSTITUT FÜR MATHEMATIK, HUMBOLDT-UNIVERSITÄT ZU BERLIN, D-10099 BERLIN, GERMANY
E-mail address: schueth@math.hu-berlin.de

DEPARTMENT OF MATHEMATICS, DARTMOUTH COLLEGE, HANOVER, NH 03755
E-mail address: david.l.webb@dartmouth.edu