# Numerical Evaluation of Approximation Methods in Stochastic Programming

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# Stochastic Programming

Stochastic Programming model:

$$v(\xi) := \min_{x \in X} \varphi(x, \xi) \qquad \left(= \min_{x \in X} \mathbb{E}_\xi[f(x, \xi)]\right)$$
$$S(\xi) := \{x \in X : \varphi(x, \xi) = v(\xi)\}$$

- analytic solutions are rarely available

$\Rightarrow$ numerical approach: discretize or approximate the stochastic process $\xi$

How to measure the quality of approximation methods?

# Approximation quality

## Theoretical results

- perturbation and stability analysis (Römisch, Schultz, ...)
- statistical estimates and bounds (Shapiro, Sen, ...)
- convergence of optimal values and/or solutions (sets) for specific approximation techniques (Pennanen, Kuhn, Pflug, ...)

# Approximation quality

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but...

- require regularity and boundedness assumptions (e.g., relatively complete recourse)
- asymptotical results or unknown (Lipschitz) constants

# Numerical Evaluation

## Evaluate approximation quality numerically?

- deviation from *real* optimal value: $|v(\xi) - v(\tilde{\xi})|$?
  - $v(\xi)$ unknown
  - Comparing $v(\tilde{\xi})$ and $v(\tilde{\tilde{\xi}})$? No.

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- deviation from *real* optimal solution set: $d(S(\xi), S(\tilde{\xi}))$ ?
  - $S(\xi)$ unknown

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  - $v(\xi)$ unknown
  - Comparing $v(\tilde{\xi})$ and $v(\tilde{\tilde{\xi}})$? No.
- deviation from *real* optimal solution set: $d(S(\xi), S(\tilde{\xi}))$ ?
  - $S(\xi)$ unknown
- "real-world" performance of approximated solution $\tilde{x}(\tilde{\xi})$:

$$``\varphi(\tilde{x}(\tilde{\xi}), \xi)``$$

  maybe a more reliable estimate of $v(\xi)$ than $v(\tilde{\xi})$

  compare $\varphi(\tilde{x}(\tilde{\xi}), \xi)$ and $\varphi(\tilde{\tilde{x}}(\tilde{\tilde{\xi}}), \xi)$

$\Rightarrow$ estimate $\varphi(\tilde{x}(\tilde{\xi}), \xi)$ by out-of-sample simulations

# Out-of-sample (oos) simulations

Estimate $\varphi(\tilde{x}(\tilde{\xi}), \xi)$ by out-of-sample (oos) simulations:

1. Approximate $\xi$ by $\tilde{\xi}$
2. Solve an approximated problem using $\tilde{\xi}$
   $\Rightarrow$ optimal value $v(\tilde{\xi})$ and optimal solution $\tilde{x}(\tilde{\xi})$

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2. Solve an approximated problem using $\tilde{\xi}$
   $\Rightarrow$ optimal value $v(\tilde{\xi})$ and optimal solution $\tilde{x}(\tilde{\xi})$
3. Draw samples (*out-of-sample* scenarios) $\{\hat{\xi}^i, i = 1, \ldots, N\}$ from $\xi$
4. Evaluation of $\tilde{x}(\tilde{\xi})$ along $\hat{\xi}^i$ yields $f(\tilde{x}, \hat{\xi}^i)$, $i = 1, \ldots, N$

$$\varphi(\tilde{x}(\tilde{\xi}), \xi) \approx \frac{1}{N} \sum_{i=1}^{N} f(\tilde{x}, \hat{\xi}^i)$$

# Evaluation of $\tilde{x}(\tilde{\xi})$ along $\{\hat{\xi}^i, i = 1, \ldots, N\}$: Literature

## Kaut & Wallace (2007)

"Evaluation of scenario-generation methods for stochastic programming"

- one-period portfolio optimization problem

# Evaluation of $\tilde{x}(\tilde{\xi})$ along $\{\hat{\xi}^i, i = 1, \ldots, N\}$: Literature

## Kaut & Wallace (2007)

"Evaluation of scenario-generation methods for stochastic programming"

- one-period portfolio optimization problem

## Chiralaksanakul & Morton (2004)

"Assessing policy quality in multi-stage stochastic programming"

## Hilli & Pennanen (2006)

"Numerical study of discretizations of multistage stochastic programs"

- multistage stochastic programs (time interval $[1, \ldots, T]$)
- given solution $\tilde{x}(\tilde{\xi})$ and oos scenarios $\hat{\xi}^i$, $i = 1, \ldots, N$
- for $i = 1, \ldots, N$: let $\hat{x}_1^i := \tilde{x}_1(\tilde{\xi})$
  for $t = 2, \ldots, T$:
  - construct tree for $[t, \ldots, T]$ conditional on $(\xi_1, ..., \xi_{t-1}) = (\hat{\xi}_1^i, ..., \hat{\xi}_{t-1}^i)$
  - solve optimization problem on $[t, \ldots, T]$ w.r.t. $(\hat{x}_1^i, \ldots, \hat{x}_{t-1}^i) \Rightarrow \hat{x}_t^i$
- numerically demanding ($\Rightarrow T$ small)
- assume relatively complete recourse

# Overview

# Setting

Multistage linear stochastic programs with or without relatively complete recourse

$$
\begin{aligned}
v(\xi) = \qquad \min \quad & \mathbb{E}\left[\sum_{t=1}^{T} \langle c_t(\xi_t), x_t \rangle\right] \\
\text{such that} \quad & x_t \geq 0, \qquad x_t \in \mathcal{F}_t, \qquad && t = 1, \ldots, T \\
& A_{t,0} x_t + A_{t,1} x_{t-1} = h_t(\xi_t), \qquad && t = 2, \ldots, T
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$$

# Evaluation of $\tilde{x}(\tilde{\xi})$ along $\{\hat{\xi}^i, i = 1, \ldots, N\}$

Given: scenario tree approximation $\tilde{\xi}$ of $\xi$ and solution $\tilde{x}(\tilde{\xi})$

Consider an out-of-sample scenario $\hat{\xi}^i$

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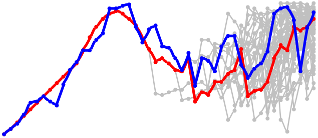

1. find a reference scenario $\tilde{\xi}^{h(i)}$ of $\tilde{\xi}$ close to $\hat{\xi}^i$ (nonanticipativity!): For $t = 2, \dots, T$:

$$\tilde{\xi}_t^{h(i)} := \operatorname{argmin}_{\tilde{\xi}_t} \|\tilde{\xi}_t - \hat{\xi}_t^i\|$$

$$\text{such that } (\tilde{\xi}_1, \dots, \tilde{\xi}_{t-1}) = (\xi_1^{h(i)}, \dots, \xi_{t-1}^{h(i)})$$

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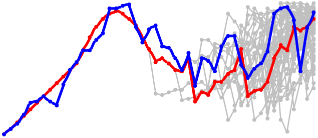

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2. consider the reference solution $\tilde{x}^i := \tilde{x}(\tilde{\xi}^{h(i)})$ along $\tilde{\xi}^{h(i)}$
3. modify $\tilde{x}^i$ to a feasible solution along $\hat{\xi}^i$ (if possible) $\Rightarrow$ oos solution $\hat{x}^i$

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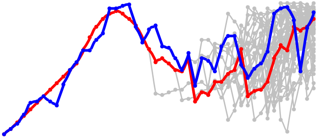

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$$\Rightarrow \text{oos-value} \quad \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \langle c_t(\hat{\xi}_t^i), \hat{x}_t^i \rangle$$

# Evaluation of $\tilde{x}(\tilde{\xi})$ along $\{\hat{\xi}^i, i = 1, \ldots, N\}$

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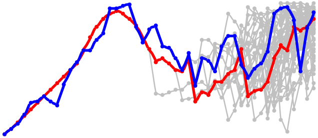

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How to adapt $\tilde{x}^i$ to a feasible and "good" solution $\hat{x}^i$ ?

$\hat{x}_1^i := \tilde{x}_1^i$. For $t = 2, \ldots, T$ (nonanticipativity!):

Find feasible $\hat{x}_t^i$ that is close to $\tilde{x}_t^i$ (and, maybe, has low costs $\langle c_t(\hat{\xi}_t^i), \hat{x}_t^i \rangle$).

# Modify $\tilde{x}^i$ to a feasible solution $\hat{x}^i$ along $\hat{\xi}^i$: Naive

Given $(\hat{x}^i_1, \dots, \hat{x}^i_{t-1})$.

Very naive approach: keep close to $\tilde{x}^i_t$

$$\begin{aligned} \hat{x}^i_t := \operatorname{argmin}_{x_t} \; & \|x_t - \tilde{x}^i_t\|_\infty \\ \text{such that } & x_t \geq 0 \\ & A_{t,0} x_t + A_{t,1} \hat{x}^i_{t-1} = h_t(\hat{\xi}^i) \end{aligned}$$

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& A_{\tau,0} x_\tau + A_{\tau,1} x_{\tau-1} \in [h_\tau^{\text{low}}, h_\tau^{\text{up}}], \qquad \tau = t+1, \ldots, T
\end{aligned}
$$

(to include at least deterministic information about the future)

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What about the costs $\langle c_t(\hat{\xi}_t^i), \hat{x}_t^i \rangle$?

# Modify $\tilde{x}^i$ to a feasible solution $\hat{x}^i$ along $\hat{\xi}^i$: Myopic

Given $(\hat{x}^i_1, \ldots, \hat{x}^i_{t-1})$.

Distance to feasible set:

$$\begin{aligned} \Delta^i_t := \min\ & \|x_t - \tilde{x}^i_t\|_\infty \\ \text{s.t. } & x_t \geq 0 \\ & A_{t,0} x_t + A_{t,1} \hat{x}^i_{t-1} = h_t(\hat{\xi}^i) \\ & A_{\tau,0} x_\tau + A_{\tau,1} x_{\tau-1} \in [h^{\text{low}}_\tau, h^{\text{up}}_\tau], \qquad \tau = t+1, \ldots, T \end{aligned}$$

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Myopic approach: keep close to $\tilde{x}_t^i$, but minimize current costs

$$\begin{aligned}\hat{x}_t^i := \operatorname{argmin}_{x_t} \ & \langle c_t(\hat{\xi}_t^i), x_t \rangle \\ \text{s.t. } & x_t \geq 0 \\ & A_{t,0} x_t + A_{t,1} \hat{x}_{t-1}^i = h_t(\hat{\xi}^i) \\ & A_{\tau,0} x_\tau + A_{\tau,1} x_{\tau-1} \in [h_\tau^{\text{low}}, h_\tau^{\text{up}}], \qquad \tau = t+1, \dots, T \\ & \|x_t - \tilde{x}_t^i\|_\infty \leq (1 + \varepsilon_{\text{rel}}) \Delta_t^i + \varepsilon_{\text{abs}}\end{aligned}$$

# Modify $\tilde{x}^i$ to a feasible solution $\hat{x}^i$ along $\hat{\xi}^i$: Myopic

Given $(\hat{x}^i_1, \dots, \hat{x}^i_{t-1})$.

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What about the future costs $Q_t(\hat{x}^i_t, \hat{\xi}^i_t)$?

## Modify $\tilde{x}^i$ to a feasible solution $\hat{x}^i$ along $\hat{\xi}^i$: Farsighted

Given $(\hat{x}_1^i, \dots, \hat{x}_{t-1}^i)$.

Distance to feasible set:

$$
\begin{aligned}
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\end{aligned}
$$

Farsighted approach: keep close to $\tilde{x}_t^i$, but minimize current and future costs

$$
\begin{aligned}
\hat{x}_t^i := \operatorname{argmin}_{x_t}\ & \langle c_t(\hat{\xi}_t^i) - \pi_{t+1}, x_t \rangle \\
\text{s.t. } & x_t \geq 0 \\
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& \|x_t - \tilde{x}_t^i\|_\infty \leq (1 + \varepsilon_{\text{rel}}) \Delta_t^i + \varepsilon_{\text{abs}}
\end{aligned}
$$

with $-\pi_{t+1}$ a subgradient of the cost-to-go function $x_t \mapsto Q_t(x_t, \tilde{\xi}_t^{h(i)})$

# Coping with lack of relatively complete recourse

Distance to feasible set

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No relatively complete recourse $\Rightarrow$ $\Delta_t^i$ may be $\infty$!

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- minimize violation of "soft constraints" (e.g., reserve requirements), proceed
- or stop ($\Rightarrow$ oos scenario is "hard infeasible")

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$\Rightarrow$ more characteristic values:

oos-value w.r.t. feasible scenarios: $\displaystyle \frac{1}{|I_{\text{feas}}|} \sum_{i \in I_{\text{feas}}} \sum_{t=1}^{T} \langle c_t(\hat{\xi}_t^i), \hat{x}_t^i \rangle$

rates of soft and hard infeas. scen.: $\displaystyle \frac{|I_{\text{soft infeas}}|}{N}, \quad \frac{|I_{\text{hard infeas}}|}{N}$

average soft infeasibility: $\displaystyle \frac{1}{|I_{\text{soft infeas}}|} \sum_{i \in I_{\text{soft infeas}}} \left( \begin{array}{c} \text{violation of} \\ \text{soft constraints} \end{array} \right)^i$

# Example: Power Scheduling

Simple power generating system:

- 3 thermal units (coal, gas & steam, gas)
- a pumped hydro unit
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$$
\begin{array}{lll}
\min & \mathbb{E}[\text{operation costs}] & \\
\text{s.t.} & \text{electricity demand is satisfied} & \forall t \\
& \text{power gradient} \leq 1/2\,(\text{operation range}) & \forall \text{units}\ \forall t \\
& \text{reserve of thermal units} \geq 1/10\,\text{demand} & \forall \text{units}\ \forall t \\
& \text{fill level of hydro storage} = 1/2\,\text{capacity} & t = 1 \\
& \text{fill level of hydro storage} \geq 1/2\,\text{capacity} & t = T
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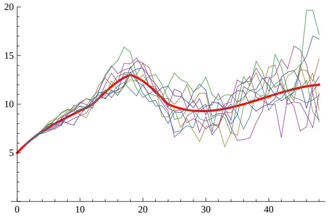
no relatively complete recourse

Time Horizon: $T = 48$ hours

# Stochastic Wind

Wind speed: model by a mean reverting autoregressive stochastic process:

$$\xi_t = \frac{3}{4}\xi_{t-1} + \mathcal{N}\left(\mu_t, \frac{1}{24}t\right)$$



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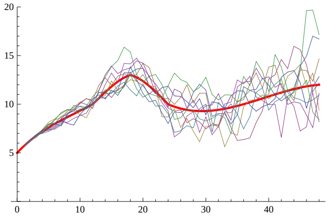
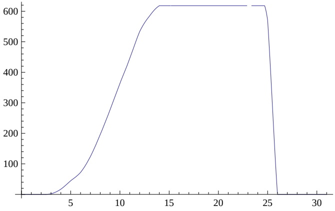
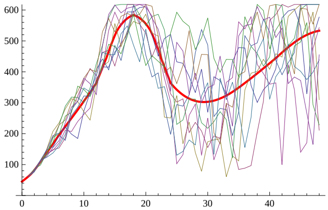
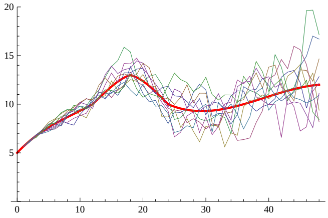



Wind energy: map wind speed to wind energy using an "aggregated power curve"



# Stochastic Wind




Wind speed: model by a mean reverting autoregressive stochastic process:

$$\xi_t = \frac{3}{4}\xi_{t-1} + \mathcal{N}\left(\mu_t, \frac{1}{24}t\right)$$

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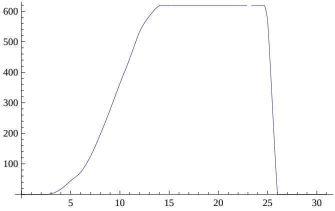


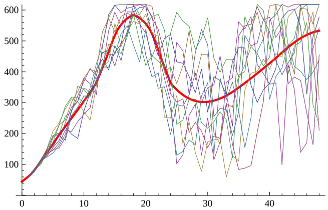


⇒ sample scenarios as input for scenario tree construction SCENRED 2.1 (H. Heitsch et.al.)

# Average distance oos scenarios to scenario trees

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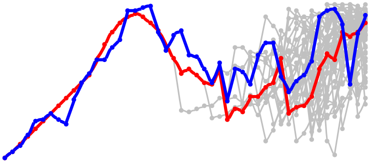

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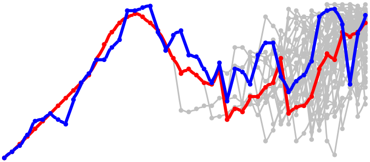

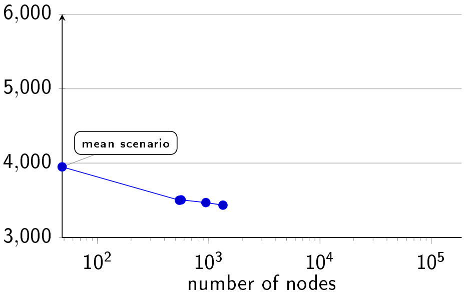

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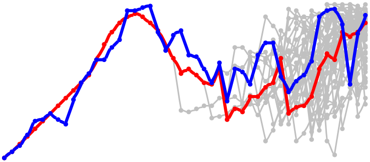

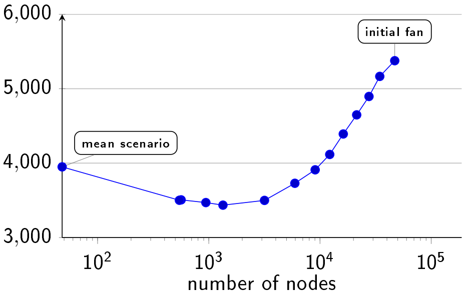

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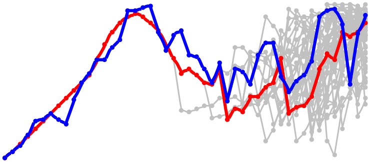

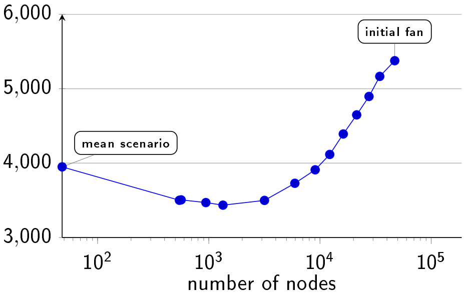


- ideal: binary branching tree 48 timestages $\Rightarrow 10^{14}$ scenarios
- $10^3$ scenarios on 48 timestages $\Rightarrow$ only $\approx 1.15$ branches in each node
- early branching leads to bad approximation of conditional distributions in later timestages (low number of initial scenarios in subtrees after a few branchings)

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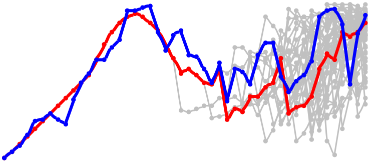

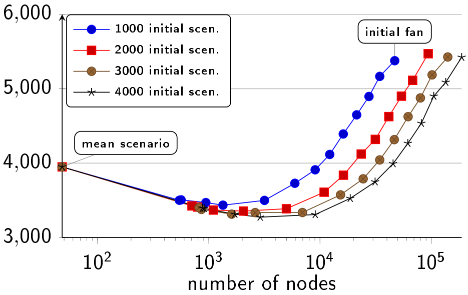


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1. construct scenario trees $\tilde{\xi}$ from 4000 scenarios
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3. modify reference solutions $\tilde{x}(\tilde{\xi}^{h(i)})$ to feasible solutions along $\hat{\xi}^i \Rightarrow \hat{x}^i$

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| naive | minimize distance $\|\tilde{x}_t(\tilde{\xi}^{h(i)}) - \hat{x}_t^i\|$ |
| myopic | minimize stage-$t$ costs within $\varepsilon$-box around $\tilde{x}_t(\tilde{\xi}^{h(i)})$ |
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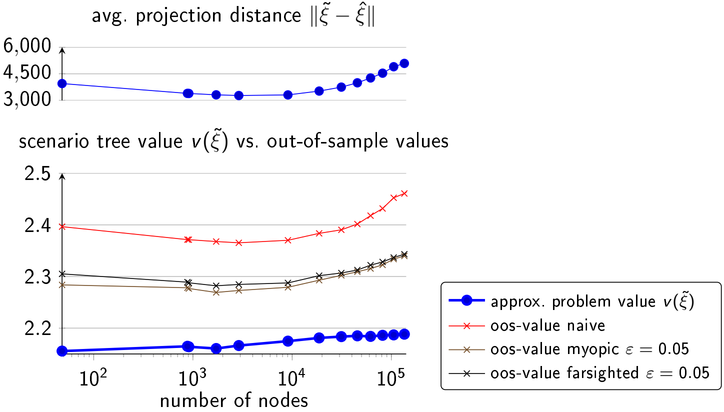

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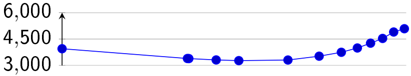
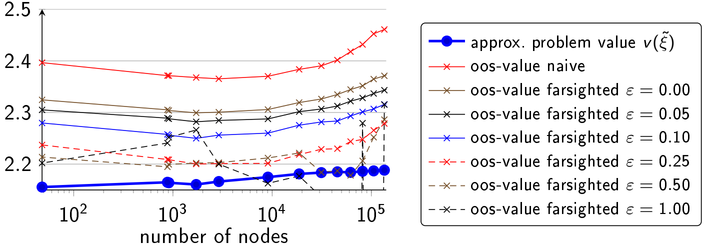



scenario tree value $v(\tilde{\xi})$ vs. out-of-sample values



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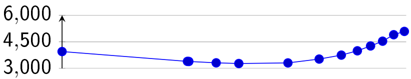
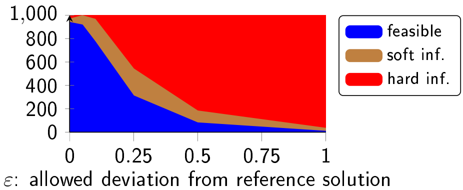



number of (soft/hard in)feasible oos-scenarios

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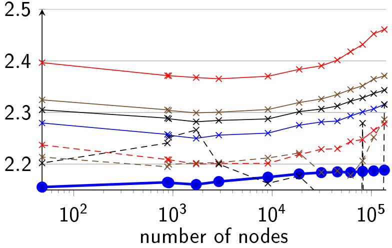


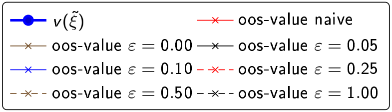

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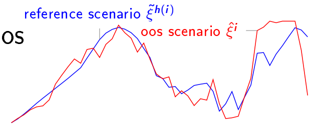
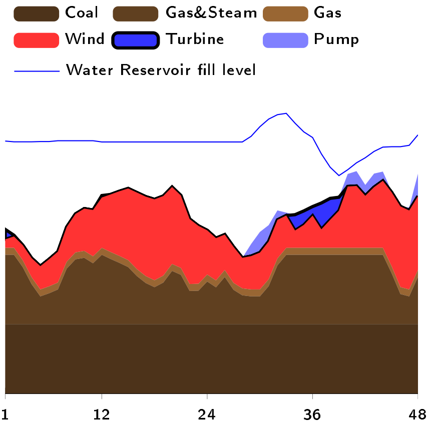



reference solution $\tilde{x}(\tilde{\xi}^{h(i)})$, cost: 2.241



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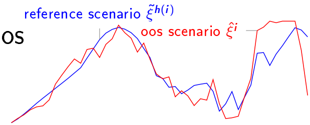


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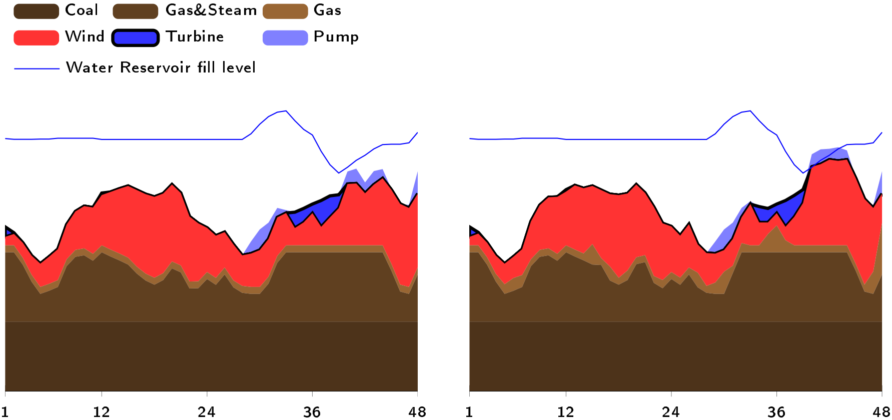

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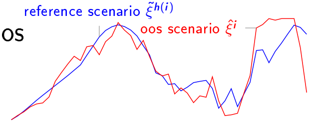


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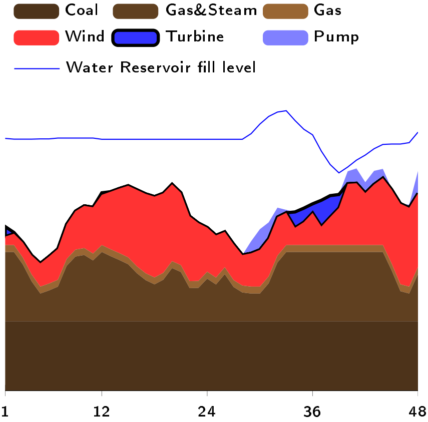


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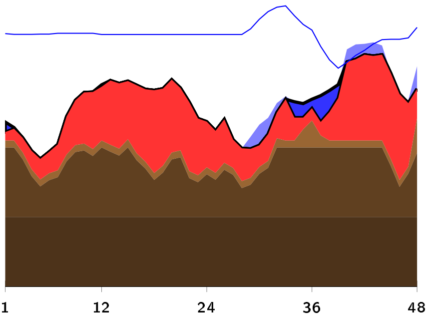

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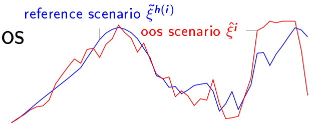


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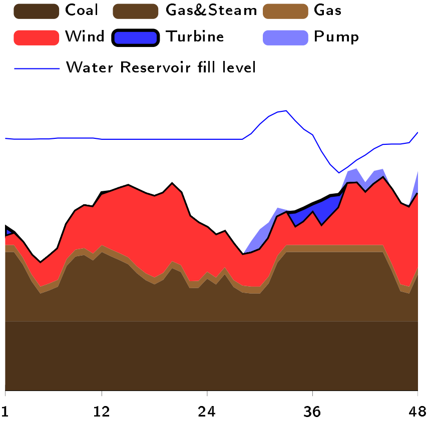


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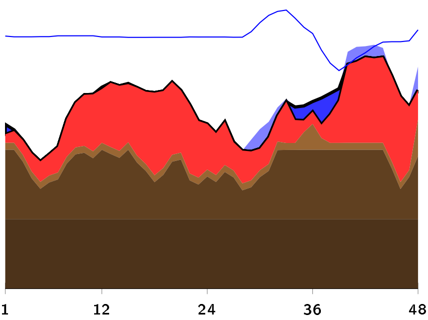

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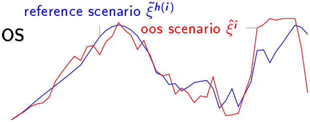


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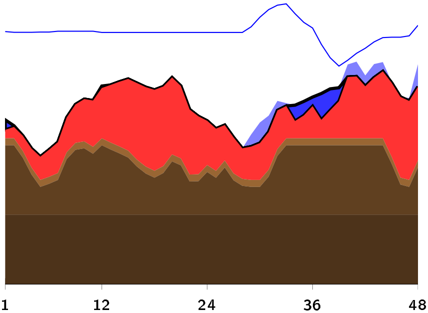
Coal | Gas&Steam | Gas
Wind | Turbine | Pump
Water Reservoir fill level

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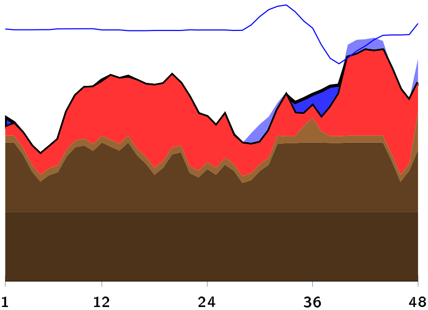

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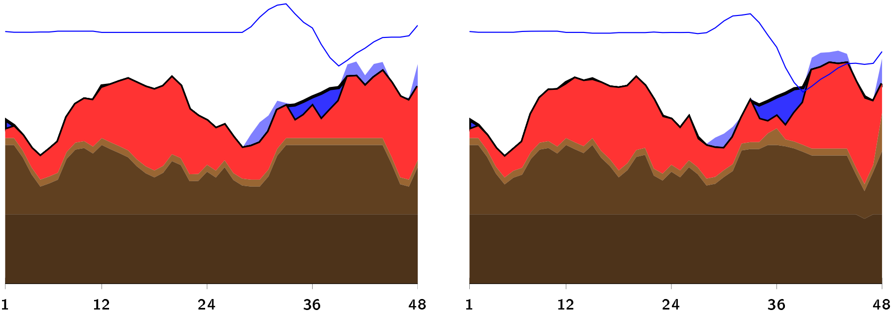

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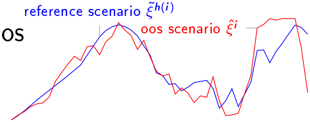


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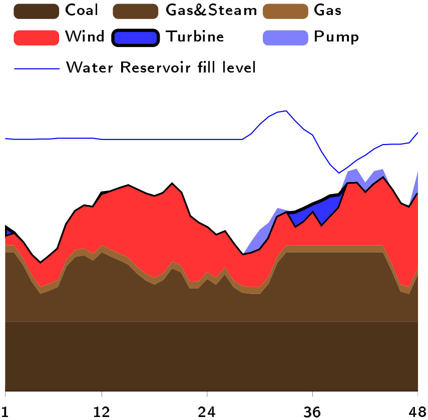


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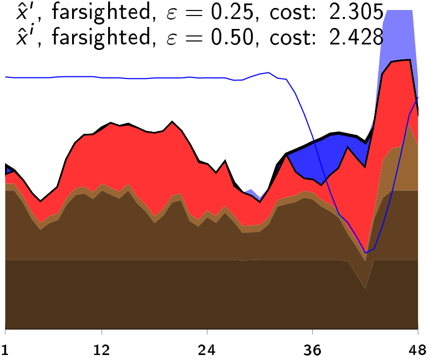
$\hat{x}^i$, farsighted, $\varepsilon = 0.50$, cost: 2.428



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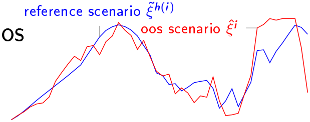


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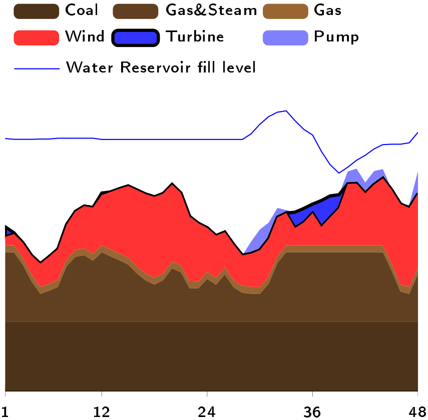


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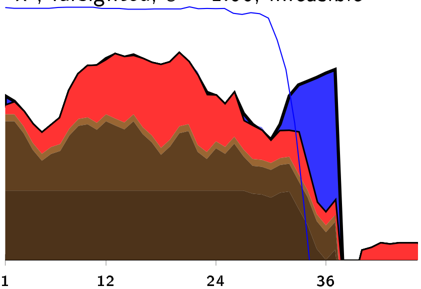
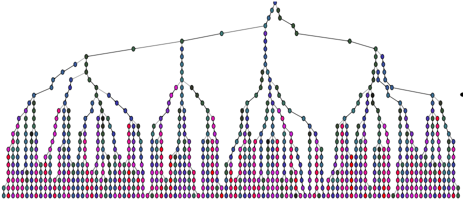
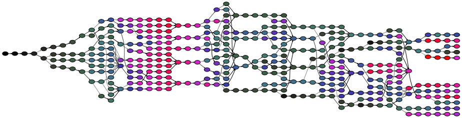


# Recombining scenario trees

- consider scenario “trees” where scenarios are allowed to recombine

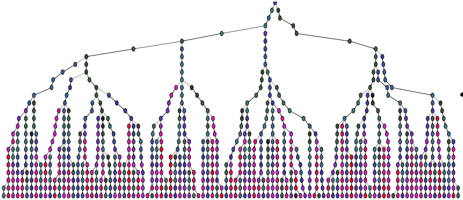


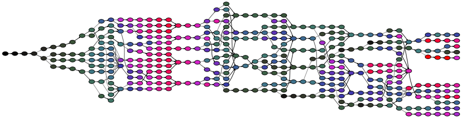


# Recombining scenario trees

- consider scenario "trees" where scenarios are allowed to recombine
- well suited for processes where $\xi_t$ depends only on short-time history

$\Rightarrow$ represent a large number of scenarios with a small number of nodes





# Recombining scenario trees

- consider scenario "trees" where scenarios are allowed to recombine
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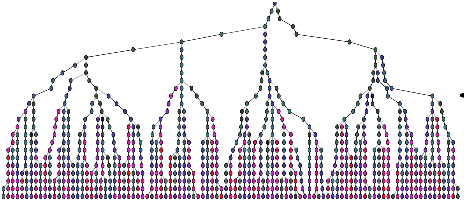
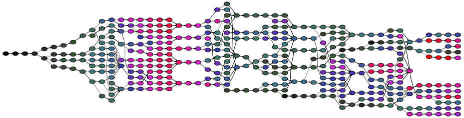
- BUT: due to time-coupling constraints, solution process is unlikely to follow same recombination pattern as scenario tree





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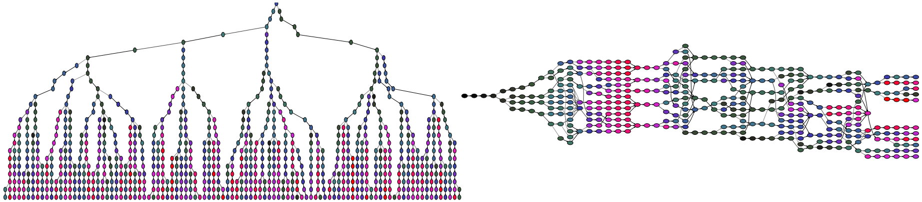
- BUT: due to time-coupling constraints, solution process is unlikely to follow same recombination pattern as scenario tree
- modification of Nested Benders decomposition allows for solution process that is dynamically recombining

  C. Küchler, S. Vigerske, Decomposition of Multistage Stochastic Programs with Recombining Scenario Trees, SPEPS 9 (2007)
- extends work of Pereira & Pinto (1991) and Infanger & Morton (1996)



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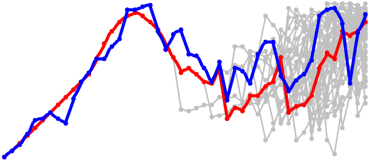

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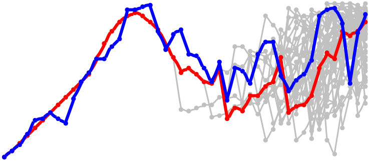

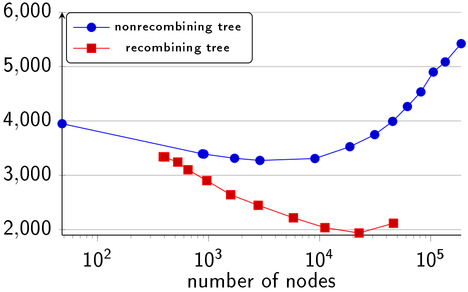

# Out-of-sample values: Non-Recomb. vs. Recomb. tree

1. construct recombining and nonrecombining scenario trees $\tilde{\xi}$
2. project 1000 oos-scenarios $\hat{\xi}^i$ onto trees: $\hat{\xi}^i \mapsto \tilde{\xi}^{h(i)}$
3. modify reference solutions $\tilde{x}(\tilde{\xi}^{h(i)})$ to feasible solutions along $\hat{\xi}^i \Rightarrow \hat{x}^i$

   farsighted minimize stage-$t$ and future costs within $\varepsilon$-box around $\tilde{x}_t(\tilde{\xi}^{h(i)})$

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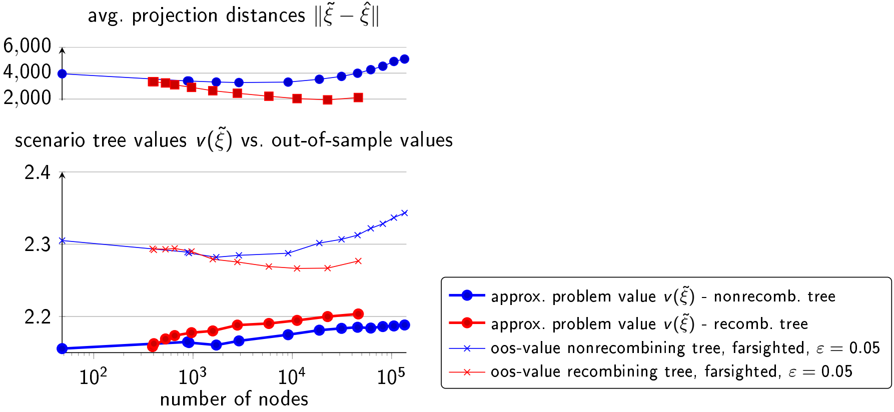

# That's it.

Thanks!

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- C. Küchler, Stability, Approximation, and Decomposition in Two- and Multistage Stochastic Programming, Ph.D. thesis, Vieweg Teubner (2009)