General Linear Methods for Integrated Circuit Design

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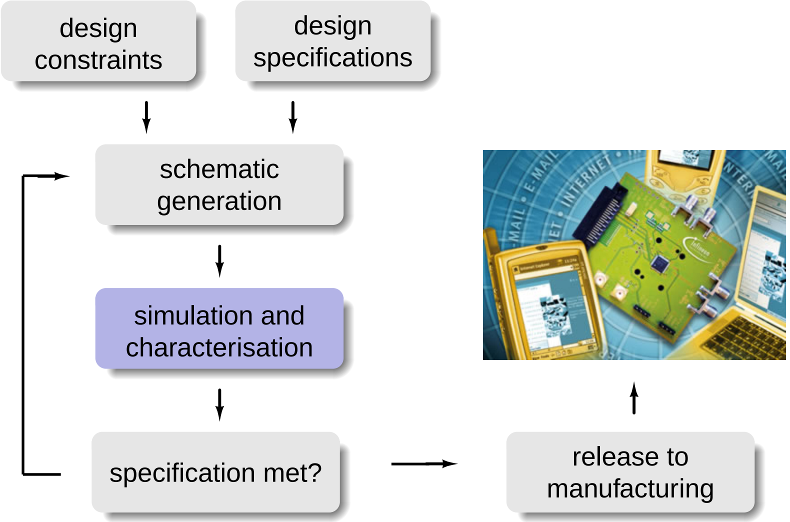
DFG Research Center MATHEON
Mathematics for key technologies

HUMBOLDT-UNIVERSITÄT ZU BERLIN

Integrated circuit design


design constraints

design specifications

schematic generation

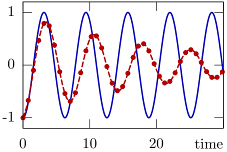
simulation and characterisation

specification met?

release to manufacturing

Classical methods

- BDF
  - artificial damping



Classical methods

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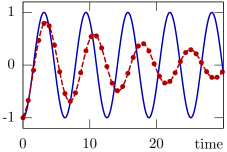
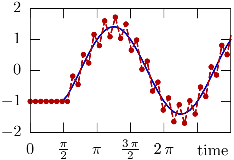


- Trapezoidal rule
    - undesired oscillations



Classical methods

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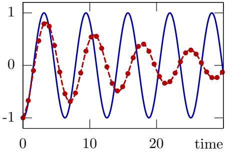

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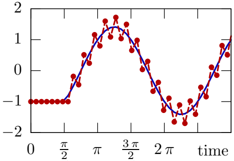
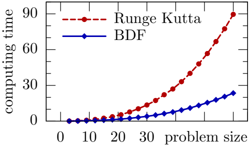
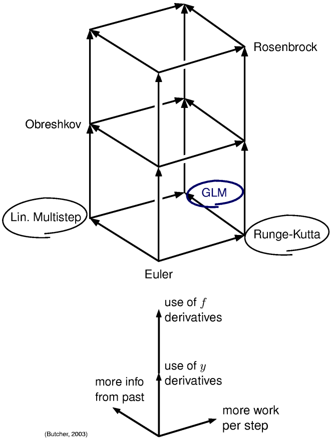


- Runge-Kutta methods
  - high computational costs



Classification of methods

- Linear multistep methods
  - low costs
  - very successful (BDF)
  - not A-stable for $p > 2$
- Runge-Kutta methods
  - very good stability properties
  - stepsize change is easy
  - high costs
- General linear methods (GLM)
  - combine advantages of both classes
  - make new methods possible
  - provide unifying framework for known methods



(Butcher, 2003)

Contents

Differential Algebraic Equations

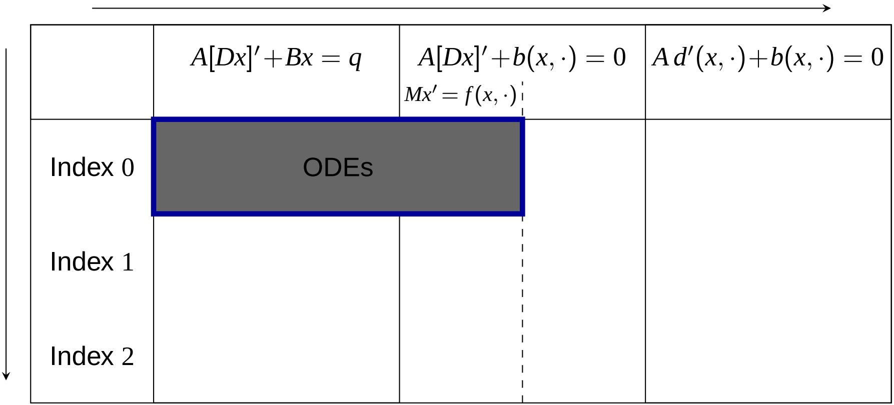
General Linear Methods

Practical General Linear Methods

DAEs of increasing complexity

	$A[Dx]'+Bx=q$	$A[Dx]'+b(x,\cdot)=0$	$A\,d'(x,\cdot)+b(x,\cdot)=0$
Index 0			
Index 1			
Index 2			

DAEs of increasing complexity

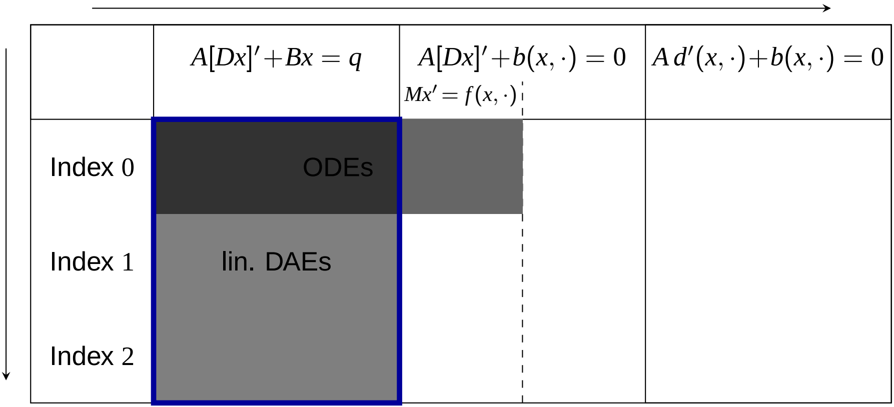


	$A[Dx]'+Bx=q$	$A[Dx]'+b(x,\cdot)=0$ $Mx'=f(x,\cdot)$	$A\,d'(x,\cdot)+b(x,\cdot)=0$
Index 0	ODEs		
Index 1			
Index 2			

$y'=f(y,t)$ ordinary differential equations

- well understood (theoretically, numerically)
- Butcher, Dahlquist, Gear, Hairer, Petzold, . . .

DAEs of increasing complexity



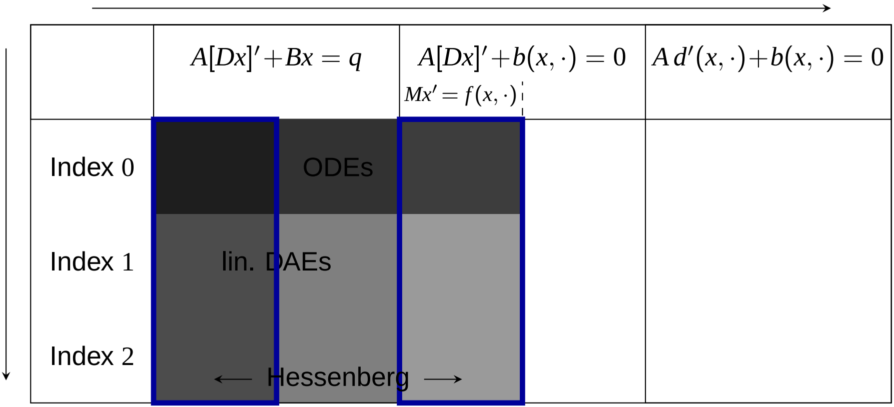
$$E\,x'+Fx=q$$

$$A[Dx]'+Bx=q$$

linear DAEs

- standard form (Hairer/Wanner, Kunkel/Mehrmann)
- prop. stated (März, Balla, Kurina)

DAEs of increasing complexity



$A[Dx]'+Bx=q$

$A[Dx]'+b(x,\cdot)=0$

$Mx'=f(x,\cdot)$

$A\,d'(x,\cdot)+b(x,\cdot)=0$

Index 0

Index 1

Index 2

ODEs

lin. DAEs

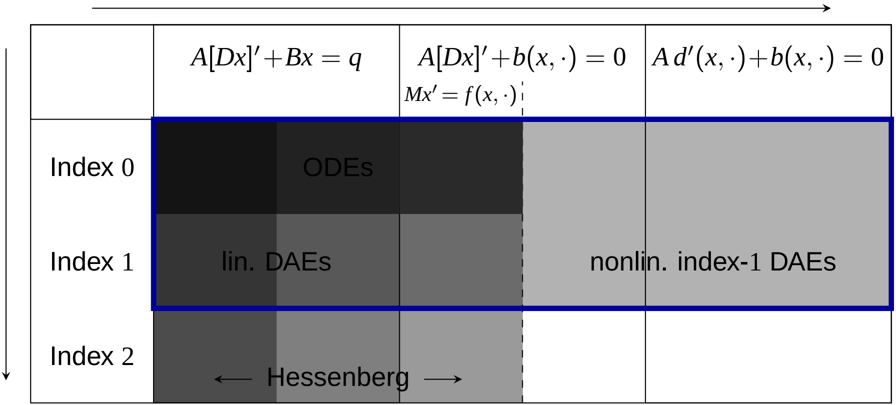
$\longleftarrow$ Hessenberg $\longrightarrow$

$$\begin{aligned} y' &= f(y,z) \\ 0 &= g(z) \end{aligned}$$

DAEs in Hessenberg form

- Runge-Kutta (Hairer/Wanner, Kværnø)
- lin. multistep (Campbell, Gear, Petzold)

DAEs of increasing complexity



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ODEs

lin. DAEs

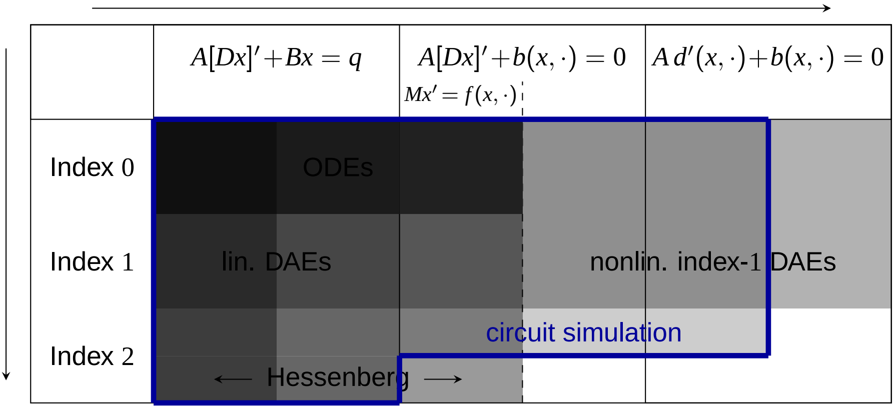
nonlin. index-1 DAEs

⟵ Hessenberg ⟶

$A\,d'(x,\cdot)+b(x,\cdot)=0$ nonlinear index-1 DAEs (prop. stated)

- extension of decoupling procedure
- März, Higueras

DAEs of increasing complexity



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ODEs

lin. DAEs

nonlin. index-1 DAEs

circuit simulation

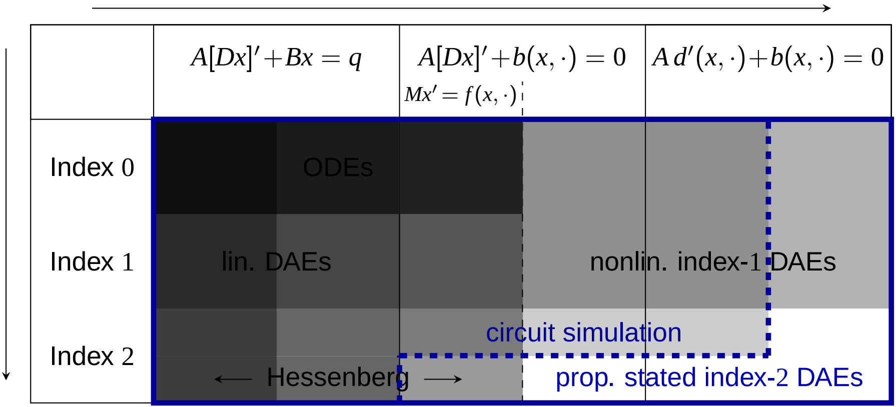
⟵ Hessenberg ⟶

$$A\,d'(x,\cdot)+b(Ux,\cdot)\\ +\mathcal{B}Tx=0$$

DAEs appearing in electrical circuit simulation

- index-2, no Hessenberg form
- Tischendorf, Estévez Schwarz (initialisation)

DAEs of increasing complexity



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circuit simulation

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⟵ Hessenberg ⟶

prop. stated index-2 DAEs

$A\,d'(x,\cdot)+b(x,\cdot)=0$ nonlinear DAEs with properly stated leading terms

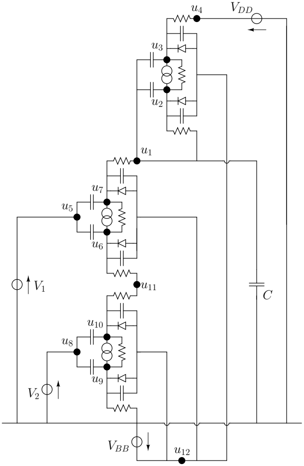
- existence and uniqueness of solutions
- convergence results for numerical methods

DAEs in electrical circuit simulation

- Modified Nodal Analysis

$$A\,\dot{q}(x,\cdot)+b(x,\cdot)=0$$

- analysis: tractability index
  - ▷ low smoothness requirements
  - ▷ use projectors (P_i, Q_i, U, T, $\ldots$) and matrix sequences

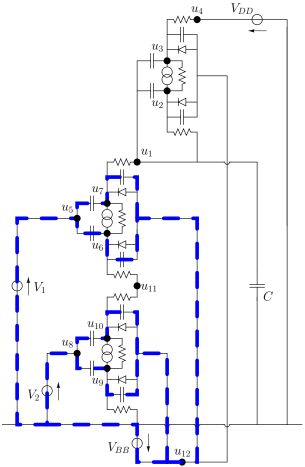


DAEs in electrical circuit simulation

- Modified Nodal Analysis

$$A\,\dot{q}(x,\cdot)+b(x,\cdot)=0$$

- analysis: tractability index
  - low smoothness requirements
  - use projectors (P_i, Q_i, U, T, $\ldots$) and matrix sequences
- index can be determined topologically
  - look for CV loops and LI cutsets
- index-2 components Tx are given by
  - currents of V-sources in CV loops
  - voltages of inductors and I-sources in LI cutsets



DAEs in electrical circuit simulation

- Modified Nodal Analysis

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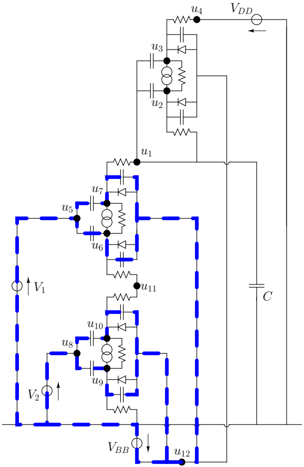
- index-2 components Tx are given by
  - currents of V-sources in CV loops
  - voltages of inductors and I-sources in LI cutsets
- index-2 components Tx enter linearly

$$A\,\dot{q}(x,\cdot)+b(Ux,\cdot)+\mathfrak{B}Tx=0$$



Estévez Schwarz (2000) *Cons. initialization for index-2 DAEs and it's application to circuit simulation*, PhD thesis

DAEs in electrical circuit simulation (cont.)

$A[Dx]' + b(Ux, \cdot) + \mathfrak{B}Tx = 0$ ➪ index-2 components enter linearly

DAEs in electrical circuit simulation (cont.)

$A[Dx]' + b(Ux,\cdot) + \mathfrak{B}Tx = 0$ ➪ index-2 components enter linearly

- **Idea:** Introduce new variables

$$u = DP_1x, \qquad w = P_1D^-(Dx)' + (Q_0 + Q_1)x.$$

For a solution x this implies

$$x = D^-u + (P_0Q_1 + Q_0P_1)w + Q_0Q_1D^-(Dx)'.$$

DAEs in electrical circuit simulation (cont.)

$A[Dx]' + b(Ux,\cdot) + \mathfrak{B}Tx = 0$ ➪ index-2 components enter linearly

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- **Consequences:**

$$Ux = D^-u + (P_0Q_1 + UQ_0)w, \quad A[Dx]' + \mathfrak{B}Tx = (AD + \mathfrak{B}T)w$$

➪ $F(u,w,\cdot) = A[Dx]' + b(Ux,\cdot) + \mathfrak{B}Tx = 0.$

Local existence and uniqueness of solutions

Theorem.

- The properly stated index-2 DAE

$$F(u,w,\cdot) = A[Dx]' + b(Ux,\cdot) + \mathfrak{B}Tx = 0, \qquad F(u_0,w_0,t_0) = 0,$$

is locally equivalent to $\mathbb{w}(u_0,t_0) = w_0, \quad F\big(u,\mathbb{w}(u,t),t\big) = 0.$

V. (2004) *General linear methods for nonlinear DAEs in circuit simulation*, SCEE

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- For every $x^0 \in \mathbb{R}^m$, the initial value problem

$$A[Dx]' + b(Ux,\cdot) + \mathfrak{B}Tx = 0, \qquad DP_1x(t_0) = DP_1x^0.$$

  is uniquely solvable. The solution $x = D^- u + z_0 + z_1$ satisfies

$$u' = f\big(u,\mathbb{w}(u,t),t\big), \quad z_1 = P_0Q_1\mathbb{w}(u,t), \quad z_0 = g\big(u,(Dz_1)',t\big).$$

V. (2004) *General linear methods for nonlinear DAEs in circuit simulation*, SCEE

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⇩

inherent ordinary differential equation

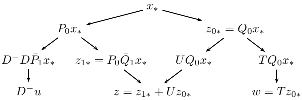
V. (2004) *General linear methods for nonlinear DAEs in circuit simulation*, SCEE

Properly stated index-2 DAEs

$$A\,\dot{q}(x,\cdot)+b(x,\cdot)=0$$

- split solution into characteristic parts

  $$x = D^{-}u + z + w$$



$$\begin{tikzcd}
 & & & x_* \arrow[dll] \arrow[drr] & & & \\
 & P_0 x_* \arrow[dl] \arrow[dr] & & & & z_{0*} = Q_0 x_* \arrow[dl] \arrow[dr] & \\
D^{-} D \bar{P}_1 x_* \arrow[d] & & z_{1*} = P_0 \bar{Q}_1 x_* \arrow[dr] & & U Q_0 x_* \arrow[dl] & & T Q_0 x_* \arrow[d] \\
D^{-} u & & & z = z_{1*} + U z_{0*} & & & w = T z_{0*}
\end{tikzcd}$$

Properly stated index-2 DAEs

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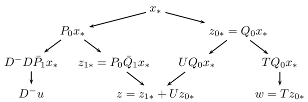
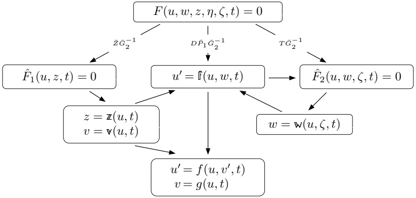


- split equations similarly

  $$\begin{aligned} u' &= f(u,v',t) & \qquad z &= \mathbb{z}(u,t) \\ v &= g(u,t) & \qquad w &= \mathbb{w}(u,v',t) \end{aligned}$$

  ▷ $x = D^{-}u + \mathbb{z}(u,\cdot) + \mathbb{w}(u,v',\cdot)$



V. (2004) *Accessible criteria for the local existence and uniqueness of DAE solutions*, MATHEON

Properly stated index-2 DAEs

$$A\,\dot q(x,\cdot)+b(x,\cdot)=0$$

- split solution into characteristic parts

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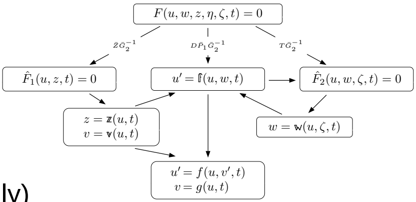
 & P_0x_* \arrow[dl] \arrow[dr] & & & & z_{0*}=Q_0x_* \arrow[dl] \arrow[dr] & \\
D^{-}D\bar P_1x_* \arrow[d] & & z_{1*}=P_0\bar Q_1x_* \arrow[dr] & & UQ_0x_* \arrow[dl] & & TQ_0x_* \arrow[d] \\
D^{-}u & & & z=z_{1*}+Uz_{0*} & & & w=Tz_{0*}
\end{tikzcd}$$

- split equations similarly

  $$\boxed{\begin{aligned} u' &= f(u,v',t) \\ v &= g(u,t) \end{aligned}} \qquad \begin{aligned} z &= \mathbb{z}(u,t) \\ w &= \mathbb{w}(u,v',t) \end{aligned}$$

  - ▷ $x\ = D^{-}u + \mathbb{z}(u,\cdot) + \mathbb{w}(u,v',\cdot)$

- $I - f_{v'}g_u$ remains non-singular (locally)



$F(u,w,z,\eta,\zeta,t)=0$

$\bar Z\bar G_2^{-1}$

$D\bar P_1\bar G_2^{-1}$

$T\bar G_2^{-1}$

$\hat F_1(u,z,t)=0$

$u'=\mathbb{f}(u,w,t)$

$\hat F_2(u,w,\zeta,t)=0$

$z=\mathbb{z}(u,t)$
$v=\mathbb{v}(u,t)$

$w=\mathbb{w}(u,\zeta,t)$

$u'=f(u,v',t)$
$v=g(u,t)$

Implicit Index-1 System

V. (2004) *Accessible criteria for the local existence and uniqueness of DAE solutions*, MATHEON

Properly stated index-2 DAEs (cont.)

$$A\,\dot{q}(x,\cdot)+b(x,\cdot)=0 \qquad \Leftrightarrow \qquad \begin{array}{ll} u'=f(u,v',t) & \qquad z=\mathbb{z}(u,t) \\ v=g(u,t) & \qquad w=\mathbb{w}(u,v',t) \end{array}$$

$$x=D^{-}u+\mathbb{z}(u,\cdot)+\mathbb{w}(u,v',\cdot)$$

V. (2006) *General linear methods for integrated circuit design*, PhD thesis

Properly stated index-2 DAEs (cont.)

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$$x = D^{-}u + \mathbb{z}(u,\cdot) + \mathbb{w}(u,v',\cdot)$$

- new decoupling procedure
- existence and uniqueness results
- only mild smoothness assumptions
- covers/extends results on
  - linear DAEs (Balla, März, Kurina)
  - nonlinear index-1 DAEs (Higueras, März)
  - DAEs $A[Dx]' + b(Ux,\cdot) + \mathfrak{B}Tx = 0$ (Tischendorf, Estévez Schwarz)
  - Hessenberg DAEs (Hairer, Lubich, Roche, Wanner)

V. (2006) *General linear methods for integrated circuit design*, PhD thesis

Decoupling and discretisation

$$\begin{tikzcd}
\begin{array}{c} A\,\dot{q}(x,t) + b(x,t) = 0 \\ \text{index-2 DAE} \end{array} \arrow[r, "\text{decoupling}"] \arrow[d, "\text{discretisation}"'] & \begin{array}{c} u' = f(u,v',t) \\ v = g(u,t) \\ + \text{ constraints} \end{array} \arrow[d, "\text{discretisation}"] \\
\text{discretised index-2 DAE} \arrow[r, leftrightarrow, dashed, "\text{decoupling}"] & \begin{array}{c} \text{discretised index-1 DAE} \\ + \text{ discretised constraints} \end{array}
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\end{tikzcd}$$

- If two subspaces associated with the DAE, DN_1 and DS_1 are constant, then this diagram commutes.
- It is always assumed that $N_0 \cap S_0$ does not depend on x.

Decoupling and discretisation

$$\begin{tikzcd}
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GLMs for ODEs $y' = f(y)$

- Linear multistep: $y_n \quad = \quad h\,\beta_0 f(y_n) \quad + \quad \sum_{i=1}^{k} \alpha_i y_{n-i}$

GLMs for ODEs $y' = f(y)$

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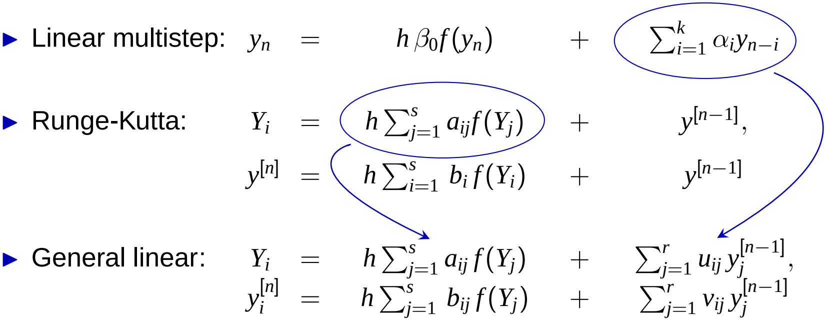
GLMs for ODEs $y' = f(y)$

- Linear multistep: $y_n = h\,\beta_0 f(y_n) + \boxed{\sum_{i=1}^{k} \alpha_i y_{n-i}}$
- Runge-Kutta:

$$\begin{aligned} Y_i &= \boxed{h \sum_{j=1}^{s} a_{ij} f(Y_j)} + y^{[n-1]}, \\ y^{[n]} &= h \sum_{i=1}^{s} b_i\, f(Y_i) + y^{[n-1]} \end{aligned}$$

GLMs for ODEs $y' = f(y)$



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- General linear:
$$\begin{aligned} Y_i &= h\sum_{j=1}^{s} a_{ij}\, f(Y_j) + \sum_{j=1}^{r} u_{ij}\, y_j^{[n-1]}, \\ y_i^{[n]} &= h\sum_{j=1}^{s} b_{ij}\, f(Y_j) + \sum_{j=1}^{r} v_{ij}\, y_j^{[n-1]} \end{aligned}$$

GLMs for ODEs $y' = f(y)$

- Linear multistep: $$y_n = h\,\beta_0 f(y_n) + \sum_{i=1}^{k} \alpha_i y_{n-i}$$
- Runge-Kutta: $$\begin{aligned} Y_i &= h\sum_{j=1}^{s} a_{ij} f(Y_j) + y^{[n-1]}, \\ y^{[n]} &= h\sum_{i=1}^{s} b_i f(Y_i) + y^{[n-1]} \end{aligned}$$
- General linear: $$\begin{aligned} Y_i &= h\sum_{j=1}^{s} a_{ij} f(Y_j) + \sum_{j=1}^{r} u_{ij}\, y_j^{[n-1]}, \\ y_i^{[n]} &= h\sum_{j=1}^{s} b_{ij} f(Y_j) + \sum_{j=1}^{r} v_{ij}\, y_j^{[n-1]} \end{aligned}$$

$$\begin{bmatrix} y_1^{[n-1]} \\ \vdots \\ y_r^{[n-1]} \end{bmatrix} \longrightarrow \boxed{\left[\begin{array}{c} Y \\ \hline y^{[n]} \end{array}\right] = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{U} \\ \hline \mathcal{B} & \mathcal{V} \end{array}\right] \cdot \left[\begin{array}{c} hf(Y) \\ \hline y^{[n-1]} \end{array}\right]} \longrightarrow \begin{bmatrix} y_1^{[n]} \\ \vdots \\ y_r^{[n]} \end{bmatrix}$$

An example method

$$\left[\begin{array}{c|c} \mathcal{A} & \mathcal{U} \\ \hline \mathcal{B} & \mathcal{V} \end{array}\right] = \left[\begin{array}{cccc|cccc}
\frac{1}{4} & 0 & 0 & 0 & 1 & 0 & -\frac{1}{32} & -\frac{1}{192} \\
\frac{49}{25} & \frac{1}{4} & 0 & 0 & 1 & -\frac{171}{100} & -\frac{49}{100} & -\frac{43}{600} \\
\frac{123}{1225} & -\frac{225}{392} & \frac{1}{4} & 0 & 1 & \frac{1363}{1400} & \frac{13941}{39200} & \frac{5379}{78400} \\
-\frac{95}{84} & -\frac{59}{84} & \frac{7}{36} & \frac{1}{4} & 1 & \frac{43}{18} & \frac{31}{42} & \frac{37}{336} \\
\hline
-\frac{95}{84} & -\frac{59}{84} & \frac{7}{36} & \frac{1}{4} & 1 & \frac{43}{18} & \frac{31}{42} & \frac{37}{336} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-\frac{268}{21} & \frac{86}{21} & -\frac{28}{9} & 4 & 0 & \frac{70}{9} & \frac{10}{21} & -\frac{5}{21} \\
-\frac{32}{21} & \frac{88}{21} & -\frac{224}{9} & 16 & 0 & \frac{56}{9} & \frac{20}{21} & -\frac{10}{21}
\end{array}\right]$$

(Butcher, 2004)

- diagonally implicit
- order $p = 3$ and stage order $q = 3$
- stiffly accurate, A-stable, L-stable
- Nordsieck form, $y_{i+1}^{[n]} \approx h^i y^{(i)}(t_n)$

An example method

$$\left[\begin{array}{c|c} \mathcal{A} & \mathcal{U} \\ \hline \mathcal{B} & \mathcal{V} \end{array}\right] = \left[\begin{array}{cccc|cccc} \frac{1}{4} & 0 & 0 & 0 & 1 & 0 & -\frac{1}{32} & -\frac{1}{192} \\ \frac{49}{25} & \frac{1}{4} & 0 & 0 & 1 & -\frac{171}{100} & -\frac{49}{100} & -\frac{43}{600} \\ \frac{123}{1225} & -\frac{225}{392} & \frac{1}{4} & 0 & 1 & \frac{1363}{1400} & \frac{13941}{39200} & \frac{5379}{78400} \\ -\frac{95}{84} & -\frac{59}{84} & \frac{7}{36} & \frac{1}{4} & 1 & \frac{43}{18} & \frac{31}{42} & \frac{37}{336} \\ \hline -\frac{95}{84} & -\frac{59}{84} & \frac{7}{36} & \frac{1}{4} & 1 & \frac{43}{18} & \frac{31}{42} & \frac{37}{336} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{268}{21} & \frac{86}{21} & -\frac{28}{9} & 4 & 0 & \frac{70}{9} & \frac{10}{21} & -\frac{5}{21} \\ -\frac{32}{21} & \frac{88}{21} & -\frac{224}{9} & 16 & 0 & \frac{56}{9} & \frac{20}{21} & -\frac{10}{21} \end{array}\right]$$

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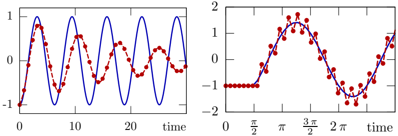
Diagonally implicit methods
with high stage order
are possible!

Why general linear methods?

$$\left[\begin{array}{c} Y \\ \hline y^{[n]} \end{array}\right] = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{U} \\ \hline \mathcal{B} & \mathcal{V} \end{array}\right] \cdot \left[\begin{array}{c} hf(Y) \\ \hline y^{[n-1]} \end{array}\right]$$

- improve stability
  - damping properties similar to RK methods



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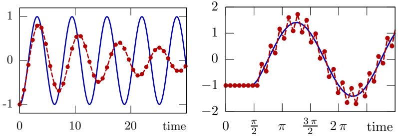

- improve efficiency
  - diagonally implicit schemes
  - solve stages sequentially

$$\mathcal{A} = \begin{bmatrix} \lambda & & 0 \\ \vdots & \ddots & \\ a_{ij} & \cdots & \lambda \end{bmatrix}$$

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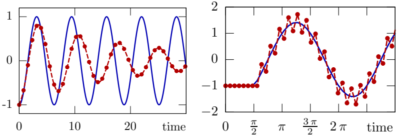

- improve efficiency
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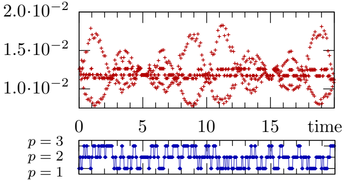
$$\mathcal{A} = \begin{bmatrix} \lambda & & 0 \\ \vdots & \ddots & \\ a_{ij} & \cdots & \lambda \end{bmatrix}$$

- benefit from high stage order
  - no order reduction
  - cheap and reliable error estimates



GLMs for index-2 DAEs

$$A\ \big(\underbrace{q\big(x(t),t\big)}\big)' + b\big(x(t),t\big) = 0$$

↑	↑	↑
singular	charges/ fluxes	voltages/ currents

GLMs for index-2 DAEs

$$A\ \big(\underbrace{q\big(x(t),t\big)}\big)' + b\big(x(t),t\big) = 0$$

↑ singular

↑ charges/ fluxes

↑ voltages/ currents

- input quantities
$$q_{k+1}^{[n-1]} \approx h^k \frac{\mathrm{d}^k}{\mathrm{d}\,t^k} q\big(x(t),t\big)$$
- $$\begin{bmatrix} q(X_n,t_c) \\ \hline q^{[n]} \end{bmatrix} = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{U} \\ \hline \mathcal{B} & \mathcal{V} \end{array}\right] \cdot \begin{bmatrix} h\,\mathcal{Q}_n' \\ \hline q^{[n-1]} \end{bmatrix}$$
  such that
  $$A\,\mathcal{Q}_n' + b(X_n,t_c) = 0$$
- solve for the stages X_n

GLMs for index-2 DAEs

$$\underset{\substack{\uparrow\\\text{singular}}}{A}\;\Big(\underbrace{q\big(x(t),t\big)}_{\substack{\uparrow\\\text{charges/}\\\text{fluxes}}}\Big)' + \underset{\substack{\uparrow\\\text{voltages/}\\\text{currents}}}{b\big(x(t),t\big)} = 0$$

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  such that
  $$A\,\mathcal{Q}_n' + b(X_n,t_c) = 0$$
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Remark

- *use implicit methods ($\mathcal{A}$ non-singular)*
- *charge conservation is guaranteed*
- *only charges / fluxes are passed on from step to step*

- *analysis uses implicit index-1 system $y' = f(y,z'), \;\; z = g(y)$*

GLMs for index-1 DAEs

Apply $\mathcal{M} = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{U} \\ \hline \mathcal{B} & \mathcal{V} \end{array}\right]$ to implicit index-1 DAEs $y' = f(y, z'), \ \ z = g(y)$

$$\begin{aligned} Y &= h\,\mathcal{A}\, f(Y, Z') + \mathcal{U}\, y^{[n]} \qquad & g(Y) &= h\,\mathcal{A}\, Z' + \mathcal{U}\, z^{[n]} \\ y^{[n+1]} &= h\,\mathcal{B}\, f(Y, Z') + \mathcal{V}\, y^{[n]} \qquad & z^{[n+1]} &= h\,\mathcal{B}\, Z' + \mathcal{V}\, z^{[n]} \end{aligned}$$

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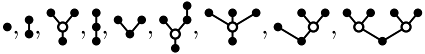
Idea:

- write numerical/exact quantities as (generalised) B-series

$$\begin{aligned} Y &= B\big(\mathbf{v};\, y(t_n), z(t_n)\big), & \qquad hZ' &= B\big(\mathbf{k};\, y(t_n), z(t_n)\big), \\ y^{[1]} &= B\big(\mathbf{u};\, y(t_n), z(t_n)\big), & \qquad z^{[1]} &= B\big(\mathbf{v};\, y(t_n), z(t_n)\big) \\ \hat{y}^{[1]} &= B\big(\mathbf{E};\, y(t_n), z(t_n)\big), & \qquad \hat{z}^{[1]} &= B\big(\mathbf{E};\, y(t_n), z(t_n)\big) \end{aligned}$$

- use Taylor series expansion to derive order conditions

$T = \Big\{ \emptyset,$  $, \dots \Big\}$

Kværnø (1990) *RK methods applied to fully implicit differential-algebraic equations of index* 1

GLMs for index-1 DAEs

Theorem. Let $\mathcal{M} = \left[\begin{array}{c|ccc} \mathcal{A} & u_1 & \cdots & u_r \\ \hline \mathcal{B} & v_1 & \cdots & v_r \end{array}\right]$ be a GLM in Nordsieck form.

- for implicit index-1 DAEs

$$\text{order } p \quad \Leftrightarrow \quad \mathbf{u}(\tau) = \mathbf{E}(\tau) \quad \forall\ \tau \in T \ \text{ with } \ |\tau| \leq p.$$

V. (2006) *General linear methods for integrated circuit design*, PhD thesis

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order conditions for $p \le 3$

order 1

	$\mathcal{B}e + v_2$	$=$	$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$

order 2

	$\mathcal{B}c + v_3$	$= \frac{1}{2}$	$\begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \\ \vdots \end{pmatrix}$
	$\mathcal{B}\mathcal{A}^{-1}(c^2 - 2u_3) + 2v_3$	$=$	$\begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \\ \vdots \end{pmatrix}$

order 3

	$\mathcal{B}c^2 + 2v_4$	$= \frac{1}{3}$
	$\mathcal{B}(\mathcal{A}c + u_3) + v_4$	$= \frac{1}{6}$
	$\mathcal{B}c\mathcal{A}^{-1}(c^2 - 2u_3) + 4v_4$	$= \frac{2}{3}$
	$\mathcal{B}(\mathcal{A}^{-1}(c^2 - 2u_3))^2 + 8v_4$	$= \frac{4}{3}$
	$\mathcal{B}\mathcal{A}^{-1}(c^3 - 6u_4) + 6v_4$	$= 1$
	$\mathcal{B}\mathcal{A}^{-1}(c\mathcal{A}c + cu_3 - 3u_4) + 3v_4$	$= \frac{1}{2}$

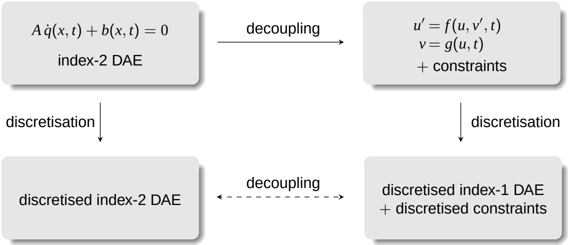
$$\begin{pmatrix} 1 \\ 3 \\ 6 \\ 6 \\ \vdots \end{pmatrix}$$

V. (2006) *General linear methods for integrated circuit design*, PhD thesis

GLMs for index-2 DAEs

$$\begin{tikzcd}
\begin{array}{c} A\,\dot{q}(x,t)+b(x,t)=0 \\ \text{index-2 DAE} \end{array} \arrow[r, "\text{decoupling}"] \arrow[d, "\text{discretisation}"'] & \begin{array}{c} u' = f(u, v', t) \\ v = g(u,t) \\ + \text{ constraints} \end{array} \arrow[d, "\text{discretisation}"] \\
\text{discretised index-2 DAE} \arrow[r, dashed, leftrightarrow, "\text{decoupling}"] & \begin{array}{c} \text{discretised index-1 DAE} \\ + \text{ discretised constraints} \end{array}
\end{tikzcd}$$

GLMs for index-2 DAEs



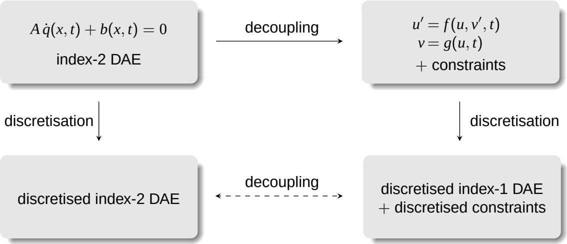
Theorem. Let $\mathcal{M}=\left[\begin{array}{c|c}\mathcal{A} & \mathcal{U}\\ \hline \mathcal{B} & \mathcal{V}\end{array}\right]$ be a GLM in Nordsieck form:

- order p for implicit index-1 DAEs
- $\mathcal{V}$ are power bounded (stability), $M_\infty=\mathcal{V}-\mathcal{B}\mathcal{A}^{-1}\mathcal{U}$ nilpotent
- stiff accuracy, stage order q.

$\mathcal{M}$ is convergent with order $\min(p,q)$ for $A\,\dot{q}(x,t)+b(x,t)=0$ (index-2)

V. (2006) *General linear methods for integrated circuit design*, PhD thesis

GLMs for index-2 DAEs



$A\,\dot q(x,t) + b(x,t) = 0$

index-2 DAE

decoupling

$u' = f(u,v',t)$
$v = g(u,t)$
$+$ constraints

discretisation

discretisation

discretised index-2 DAE

decoupling

discretised index-1 DAE
$+$ discretised constraints

Diagonally implicit methods with high stage order are possible!

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V. (2006) *General linear methods for integrated circuit design*, PhD thesis

Contents

Differential Algebraic Equations

General Linear Methods

Practical General Linear Methods

An order 2 method

construct a method of the type

$$\left[\begin{array}{c} Y_1 \\ Y_2 \\ \hline y_1^{[n]} \\ y_2^{[n]} \end{array}\right] = \left[\begin{array}{cc|cc} a_{11} & a_{12} & u_{11} & u_{12} \\ a_{21} & a_{22} & u_{21} & u_{22} \\ \hline b_{11} & b_{12} & v_{11} & v_{12} \\ b_{21} & b_{22} & v_{21} & v_{22} \end{array}\right] \cdot \left[\begin{array}{c} hf(Y_1) \\ hf(Y_2) \\ \hline y_1^{[n-1]} \\ y_2^{[n-1]} \end{array}\right], \quad c = \left[\begin{array}{c} c_1 \\ c_2 \end{array}\right]$$

1. $\mathcal{A}$ diagonally implicit
2. order 2 for impl. index-1
3. stage order 2
4. A-stability
5. L-stability
6. stability at 0 ($\mathcal{V}$ power bounded)
7. $M_\infty = \mathcal{V} - \mathcal{B}\mathcal{A}^{-1}\mathcal{U}$ nilpotent
8. stiff accuracy

An order 2 method

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$$\left[\begin{array}{c} Y_1 \\ Y_2 \\ \hline y_1^{[n]} \\ y_2^{[n]} \end{array}\right] = \left[\begin{array}{cc|cc} \frac{2\lambda-1}{2(\lambda-1)} & 0 & 1 & \frac{2\lambda-1}{2(\lambda-1)} \\ \frac{1-\lambda}{2} & \lambda & 1 & \frac{1-\lambda}{2} \\ \hline \frac{1-\lambda}{2} & \lambda & 1 & \frac{1-\lambda}{2} \\ 0 & 1 & 0 & 0 \end{array}\right] \cdot \left[\begin{array}{c} hf(Y_1) \\ hf(Y_2) \\ \hline y_1^{[n-1]} \\ y_2^{[n-1]} \end{array}\right], \quad c = \left[\begin{array}{c} 2\lambda \\ 1 \end{array}\right]$$

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GLIMDA – a DAE solver based on GLMs

General **LI**near **M**ethods for **D**ifferential **A**lgebraic equations

- solves DAEs $f\big(\dot{q}(x,t),\ x,\ t\big) = 0$
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Order control

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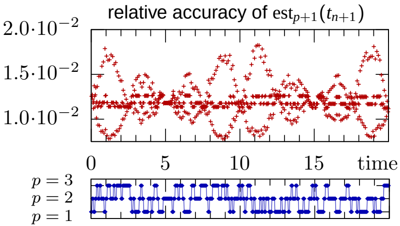
Stepsize control

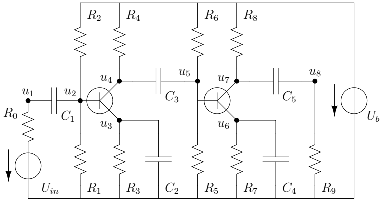
- linear combinations
  $$\mathrm{est}_{p+1}(t_{n+1}) \\ = \delta_0\, q_2^{[n]} + \delta_1\, h\, Q_1' + \cdots + \delta_s\, h\, Q_s'$$


relative accuracy of $\mathrm{est}_{p+1}(t_{n+1})$

- **test:** $x' = -\frac{1}{10}\big(x - e^{\mathrm{i}\,t}\big) + \mathrm{i}\, e^{\mathrm{i}\,t}$
  random order and fixed-variable steps $h_{n+1} = \varrho(n)\, h_n$
  Butcher, Podhaisky *On error estimation in general linear methods for stiff ODEs* (2006)

Transistor amplifier circuit (index-1)

- dimension 8
- well-known benchmark circuit
- amplification due to transistors
- `rtol` $= 10^{-j/2}, \quad j = 0, \ldots, 8$, and `atol` $= 10^{-6} \cdot$ `rtol`



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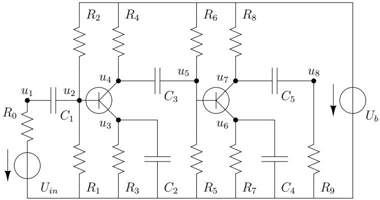
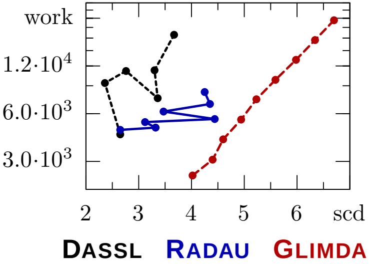
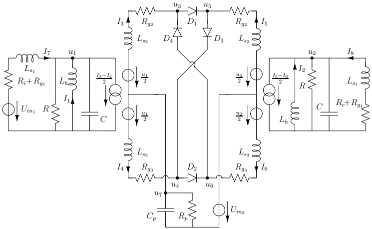


- work = # f-eval + # j-eval
- higher accuracy with less work
- almost a straight line



Ringmodulator (index-2)

- dimension 15
- low-frequent signal U_{in_1} is mixed with a high-frequent signal U_{in_2}
- $\texttt{rtol} = 10^{-j/2}, \quad j = 4, 5, \ldots, 10$ and $\texttt{atol} = 10^{-2} \cdot \texttt{rtol}$



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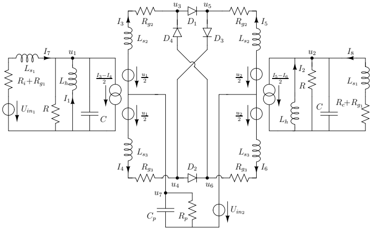

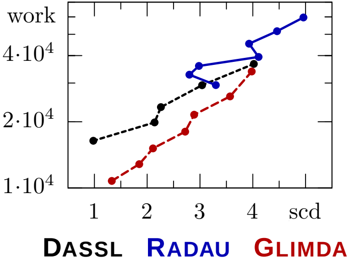

Summary

General linear methods for integrated circuit design

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- analysis of DAEs

  $$A\,\dot{q}(x,t) + b(x,t) = 0$$

  - decoupling procedure
  - exploit structure of circuit equations
  - implicit index-1 system
- analysis of GLMs
  - order conditions for implicit index-1 DAE
  - convergence via decoupling procedure
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- efficiency due to diagonally implicitness
- benefits of high stage
  - no order reduction
  - dense output
  - error estimation
- reliable solution of DAEs with properly stated leading terms