General Linear Methods for nonlinear index-2 DAEs

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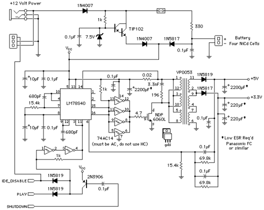
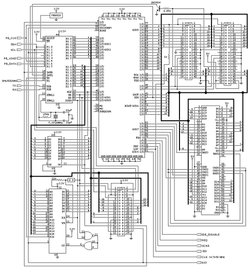
Designing an mp3-player board http://www.pjrc.com

power supply section



main processing unit



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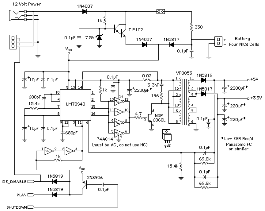
power supply section



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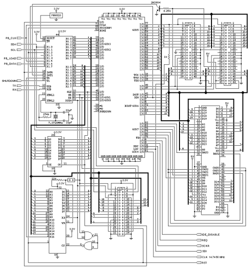
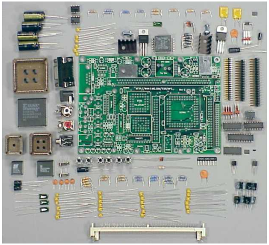
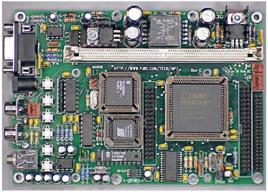


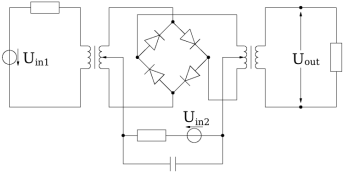


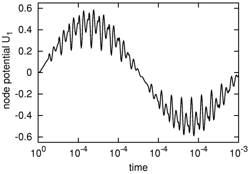


Test

Circuit equations

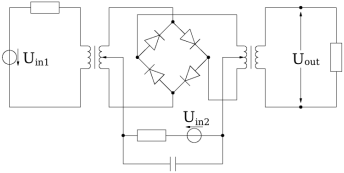
Ringmodulator (index-2)

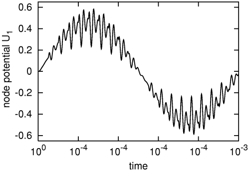




Circuit equations

Ringmodulator (index-2)



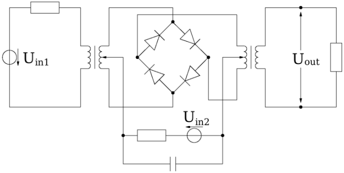


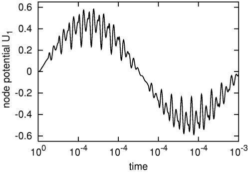
$$
\begin{aligned}
Q_1'(U_1) &= I_1 - I_3/2 + I_4/2 + I_7 - U_1/R, & \Phi_1'(I_1) &= -U_1, \\
Q_2'(U_2) &= I_2 - I_5/2 + I_6/2 + I_8 - U_2/R, & \Phi_2'(I_2) &= -U_2, \\
0 &= I_3 - d(U_{D_1}) + d(U_{D_4}), & \Phi_3'(I_3) &= U_1/2 - U_3 - R_{G_2} I_3, \\
0 &= -I_4 + d(U_{D_2}) - d(U_{D_3}), & \Phi_4'(I_4) &= -U_1/2 + U_4 - R_{G_3} I_4, \\
0 &= I_5 + d(U_{D_1}) - d(U_{D_3}), & \Phi_5'(I_5) &= U_2/2 - U_5 - R_{G_2} I_5, \\
0 &= -I_6 - d(U_{D_2}) + d(U_{D_4}), & \Phi_6'(I_6) &= -U_2/2 + U_6 - R_{G_3} I_6, \\
Q_7'(U_7) &= -U_7/R_p + d(U_{D_1}) + d(U_{D_2}) & \Phi_7'(I_7) &= -U_1, \; + U_{in\,1} \\
&\qquad - d(U_{D_3}) - d(U_{D_4}), & &\qquad - (R_J + R_{G_1}) I_7, \\
& & \Phi_8'(I_8) &= -U_2 - (R_C + R_{G_1}) I_8.
\end{aligned}
$$

- ▶ model equations generated automatically

Circuit equations

Ringmodulator (index-2)





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Q_1'(U_1) &= I_1 - I_3/2 + I_4/2 + I_7 - U_1/R,\\
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$$

$$
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\Phi_1'(I_1) &= -U_1,\\
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\Phi_3'(I_3) &= U_1/2 - U_3 - R_{G_2} I_3,\\
\Phi_4'(I_4) &= -U_1/2 + U_4 - R_{G_3} I_4,\\
\Phi_5'(I_5) &= U_2/2 - U_5 - R_{G_2} I_5,\\
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&\quad - (R_J + R_{G_1}) I_7,\\
\Phi_8'(I_8) &= -U_2 - (R_C + R_{G_1}) I_8 .
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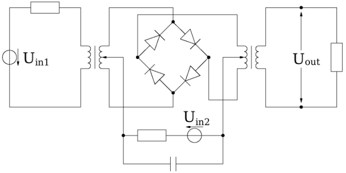
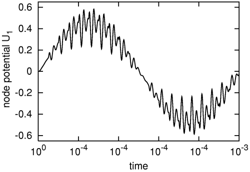
- model equations generated automatically
- differential equations

Circuit equations

Ringmodulator (index-2)





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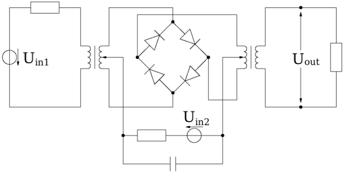
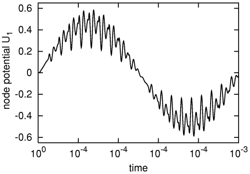
\end{aligned}$$

- model equations generated automatically
- differential equations
- algebraic equations

Circuit equations

Ringmodulator (index-2)





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\end{aligned}
$$

- model equations generated automatically
- differential equations
- algebraic equations

$$
A\,\Big(\underbrace{q\big(x(t),t\big)}\Big)' + b\big(x(t),t\big) = 0
$$

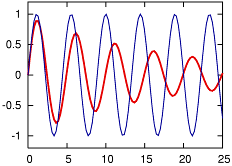
↑ singular

↑ charges/ fluxes

↑ voltages/ currents

Classical methods

- BDF
  - artificial damping



Classical methods

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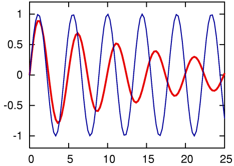
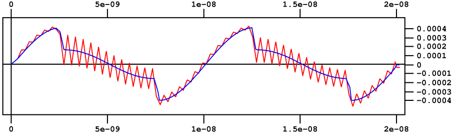


- Trapezoidal rule
  - undesired oscillations



Classical methods

- BDF
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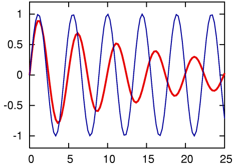

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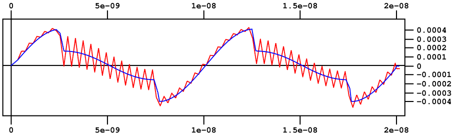
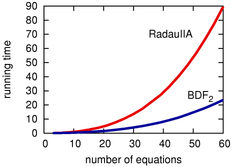
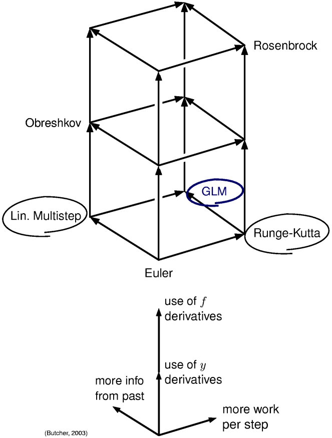


- Runge-Kutta methods
  - high computational costs



Classification of methods

- Linear multistep methods
  - low costs
  - very successful (BDF)
  - not A-stable for $p > 2$
- Runge-Kutta methods
  - very good stability properties
  - stepsize change is easy
  - high costs
- General linear methods (*GLM*)
  - combine advantages of both classes
  - make new methods possible
  - provide unifying framework for known methods



(Butcher, 2003)

Contents

GLMs for ODEs $y' = f(y)$

- Linear multistep: $y_n \quad = \qquad h\,\beta_0 f(y_n) \qquad + \qquad \sum_{i=1}^{k} \alpha_i y_{n-i}$

GLMs for ODEs $y' = f(y)$

- Linear multistep: $y_n \quad = \quad h\,\beta_0 f(y_n) \quad + \quad \boxed{\sum_{i=1}^{k} \alpha_i y_{n-i}}$

GLMs for ODEs $y' = f(y)$

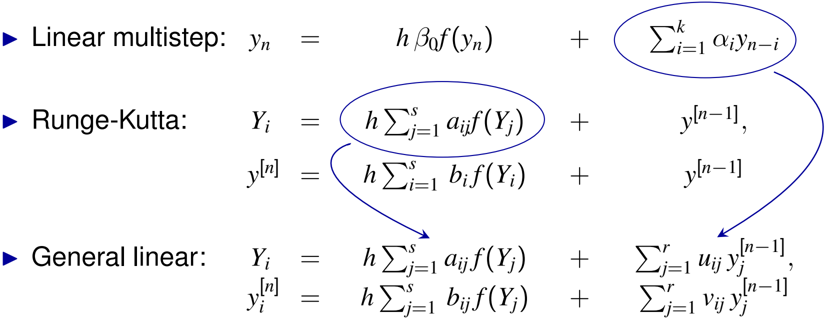
- Linear multistep: $y_n = h\,\beta_0 f(y_n) + \boxed{\sum_{i=1}^{k} \alpha_i y_{n-i}}$
- Runge-Kutta:

$$
\begin{aligned}
Y_i &= \boxed{h \sum_{j=1}^{s} a_{ij} f(Y_j)} + y^{[n-1]}, \\
y^{[n]} &= h \sum_{i=1}^{s} b_i f(Y_i) + y^{[n-1]}
\end{aligned}
$$

GLMs for ODEs $y' = f(y)$

- Linear multistep: $y_n = h\,\beta_0 f(y_n) + \sum_{i=1}^{k} \alpha_i y_{n-i}$
- Runge-Kutta:
$$\begin{aligned} Y_i &= h \sum_{j=1}^{s} a_{ij} f(Y_j) + y^{[n-1]}, \\ y^{[n]} &= h \sum_{i=1}^{s} b_i f(Y_i) + y^{[n-1]} \end{aligned}$$
- General linear:
$$\begin{aligned} Y_i &= h \sum_{j=1}^{s} a_{ij} f(Y_j) + \sum_{j=1}^{r} u_{ij}\, y_j^{[n-1]}, \\ y_i^{[n]} &= h \sum_{j=1}^{s} b_{ij} f(Y_j) + \sum_{j=1}^{r} v_{ij}\, y_j^{[n-1]} \end{aligned}$$

GLMs for ODEs $y' = f(y)$



- Linear multistep: $y_n = h\,\beta_0 f(y_n) + \sum_{i=1}^{k} \alpha_i y_{n-i}$
- Runge-Kutta: $Y_i = h\sum_{j=1}^{s} a_{ij} f(Y_j) + y^{[n-1]},$

  $y^{[n]} = h\sum_{i=1}^{s} b_i f(Y_i) + y^{[n-1]}$
- General linear: $Y_i = h\sum_{j=1}^{s} a_{ij} f(Y_j) + \sum_{j=1}^{r} u_{ij}\, y_j^{[n-1]},$

  $y_i^{[n]} = h\sum_{j=1}^{s} b_{ij} f(Y_j) + \sum_{j=1}^{r} v_{ij}\, y_j^{[n-1]}$

$$\begin{bmatrix} y_1^{[n-1]} \\ \vdots \\ y_r^{[n-1]} \end{bmatrix} \longrightarrow \boxed{\left[\begin{array}{c} Y \\ \hline y^{[n]} \end{array}\right] = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{U} \\ \hline \mathcal{B} & \mathcal{V} \end{array}\right] \cdot \left[\begin{array}{c} hf(Y) \\ \hline y^{[n-1]} \end{array}\right]} \longrightarrow \begin{bmatrix} y_1^{[n]} \\ \vdots \\ y_r^{[n]} \end{bmatrix}$$

An example method

$$\left[\begin{array}{c} Y_1 \\ Y_2 \\ Y_3 \\ \hline y_n \\ hf(y_n) \end{array}\right] = \left[\begin{array}{ccc|cc} \frac{1}{6} & 0 & 0 & 1 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & 1 & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{9} & \frac{1}{3} & 1 & \frac{1}{6} \\ \hline \frac{1}{6} & \frac{2}{9} & \frac{1}{3} & 1 & \frac{1}{6} \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \cdot \left[\begin{array}{c} hf(Y_1) \\ hf(Y_2) \\ hf(Y_3) \\ \hline y_{n-1} \\ hf(y_{n-1}) \end{array}\right] \qquad \begin{bmatrix} s = 3 \\ r = 2 \\ p = 2 \\ q = 2 \end{bmatrix}$$

An example method

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \hline y_n \\ hf(y_n) \end{bmatrix} = \left[\begin{array}{ccc|cc} \frac{1}{6} & 0 & 0 & 1 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & 1 & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{9} & \frac{1}{3} & 1 & \frac{1}{6} \\ \hline \frac{1}{6} & \frac{2}{9} & \frac{1}{3} & 1 & \frac{1}{6} \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \cdot \begin{bmatrix} hf(Y_1) \\ hf(Y_2) \\ hf(Y_3) \\ \hline y_{n-1} \\ hf(y_{n-1}) \end{bmatrix} \qquad \begin{bmatrix} s=3 \\ r=2 \\ p=2 \\ q=2 \end{bmatrix}$$

$$\begin{aligned} Y_1 &= \frac{h}{6}f(Y_1) & &+ y_{n-1} + \frac{h}{6}f(y_{n-1}) \\ Y_2 &= \frac{h}{6}f(Y_1) + \frac{h}{6}f(Y_2) & &+ y_{n-1} + \frac{h}{6}f(y_{n-1}) \\ Y_3 &= \frac{h}{6}f(Y_1) + \frac{2h}{9}f(Y_2) + \frac{h}{3}f(Y_3) & &+ y_{n-1} + \frac{h}{6}f(y_{n-1}) \end{aligned}$$

An example method

$$
\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \hline y_n \\ hf(y_n) \end{bmatrix}
=
\left[\begin{array}{ccc|cc}
\frac{1}{6} & 0 & 0 & 1 & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & 0 & 1 & \frac{1}{6} \\
\frac{1}{6} & \frac{2}{9} & \frac{1}{3} & 1 & \frac{1}{6} \\
\hline
\frac{1}{6} & \frac{2}{9} & \frac{1}{3} & 1 & \frac{1}{6} \\
0 & 0 & 1 & 0 & 0
\end{array}\right]
\cdot
\begin{bmatrix} hf(Y_1) \\ hf(Y_2) \\ hf(Y_3) \\ \hline y_{n-1} \\ hf(y_{n-1}) \end{bmatrix}
\qquad
\begin{bmatrix} s = 3 \\ r = 2 \\ p = 2 \\ q = 2 \end{bmatrix}
$$

$$
\begin{aligned}
Y_1 &= \frac{h}{6}f(Y_1) & &+ y_{n-1} + \frac{h}{6}f(y_{n-1}) = y_{n-1} + \frac{h}{6}\left(f(y_{n-1}) + f(Y_1)\right) \\
Y_2 &= \frac{h}{6}f(Y_1) + \frac{h}{6}f(Y_2) & &+ y_{n-1} + \frac{h}{6}f(y_{n-1}) = Y_1 + \frac{h}{6}\left(f(Y_1) + f(Y_2)\right) \\
Y_3 &= \frac{h}{6}f(Y_1) + \frac{2h}{9}f(Y_2) + \frac{h}{3}f(Y_3) & &+ y_{n-1} + \frac{h}{6}f(y_{n-1}) = -\frac{1}{3}Y_1 + \frac{4}{3}Y_2 + \frac{h}{3}f(Y_3)
\end{aligned}
$$

- two TR steps followed by one BDF step (stepsize $\frac{h}{3}$)

An example method

$$\left[\begin{array}{c} Y_1 \\ Y_2 \\ Y_3 \\ \hline y_n \\ hf(y_n)\end{array}\right] = \left[\begin{array}{ccc|cc} \frac{1}{6} & 0 & 0 & 1 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & 1 & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{9} & \frac{1}{3} & 1 & \frac{1}{6} \\ \hline \frac{1}{6} & \frac{2}{9} & \frac{1}{3} & 1 & \frac{1}{6} \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \cdot \left[\begin{array}{c} hf(Y_1) \\ hf(Y_2) \\ hf(Y_3) \\ \hline y_{n-1} \\ hf(y_{n-1})\end{array}\right] \qquad \begin{bmatrix} s=3 \\ r=2 \\ p=2 \\ q=2 \end{bmatrix}$$

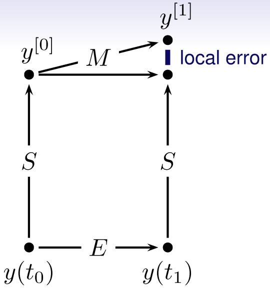
$$\begin{aligned} Y_1 &= \frac{h}{6}f(Y_1) + y_{n-1} + \frac{h}{6}f(y_{n-1}) = y_{n-1} + \frac{h}{6}\big(f(y_{n-1}) + f(Y_1)\big) \\ Y_2 &= \frac{h}{6}f(Y_1) + \frac{h}{6}f(Y_2) + y_{n-1} + \frac{h}{6}f(y_{n-1}) = Y_1 + \frac{h}{6}\big(\,f(Y_1) \;+ f(Y_2)\big) \\ Y_3 &= \frac{h}{6}f(Y_1) + \frac{2h}{9}f(Y_2) + \frac{h}{3}f(Y_3) + y_{n-1} + \frac{h}{6}f(y_{n-1}) = -\frac{1}{3}Y_1 + \frac{4}{3}Y_2 + \frac{h}{3}f(Y_3) \end{aligned}$$

- two TR steps followed by one BDF step (stepsize $\frac{h}{3}$)
- output: $y_n = Y_3$ coincides with last stage (stiff accuracy)
- input: $y_{i+1}^{[n-1]} \approx h^i y^{(i)}(t_{n-1})$ (Nordsieck form)

Order and stage order

$$\left[\begin{array}{c} Y \\ \hline y^{[n]} \end{array}\right] = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{U} \\ \hline \mathcal{B} & \mathcal{V} \end{array}\right] \cdot \left[\begin{array}{c} hf(Y) \\ \hline y^{[n-1]} \end{array}\right]$$

- let S be a starting procedure



Order and stage order

$$\left[\begin{array}{c} Y \\ \hline y^{[n]} \end{array}\right] = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{U} \\ \hline \mathcal{B} & \mathcal{V} \end{array}\right] \cdot \left[\begin{array}{c} hf(Y) \\ \hline y^{[n-1]} \end{array}\right]$$

$$\begin{tikzcd}
 & y^{[1]} \arrow[d, dash, "\text{local error}"] \\
y^{[0]} \arrow[ur, "M"] \arrow[r] & \bullet \\
y(t_0) \arrow[u, "S"] \arrow[r, "E"] & y(t_1) \arrow[u, "S"']
\end{tikzcd}$$

- let S be a starting procedure
- the method has order p, if $$y^{[1]} = \hat{y}^{[1]} + \mathcal{O}(h^{p+1})$$
- the method has stage order q, if $$Y_i = y(t_0 + c_i\, h) + \mathcal{O}(h^{q+1})$$

Order and stage order

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$$\left[\begin{array}{c} Y \\ \hline y^{[n]} \end{array}\right] = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{U} \\ \hline \mathcal{B} & \mathcal{V} \end{array}\right] \cdot \left[\begin{array}{c} hf(Y) \\ \hline y^{[n-1]} \end{array}\right]$$

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- the method has stage order q, if $\qquad Y_i = y(t_0 + c_i h) + \mathcal{O}(h^{q+1})$

Theorem (Butcher/Wright)

There are methods with $p = q$ for arbitrary order.

It is possible to choose $\mathcal{A} = \begin{bmatrix} \lambda & & \\ & \ddots & \\ * & & \lambda \end{bmatrix} \quad \Rightarrow$ *reduce costs.*

Contents

GLMs for index-2 DAEs

$$\underset{\substack{\uparrow\\ \text{singular}}}{A}\ \Big(\underbrace{q\big(x(t),t\big)}_{\substack{\uparrow\\ \text{charges/}\\ \text{fluxes}}}\Big)' + b\big(\underset{\substack{\uparrow\\ \text{voltages/}\\ \text{currents}}}{x(t)},t\big) = 0$$

GLMs for index-2 DAEs

$$\underset{\substack{\uparrow\\ \text{singular}}}{A}\ \Big(\underbrace{q\big(x(t),t\big)}_{\substack{\uparrow\\ \text{charges/}\\ \text{fluxes}}}\Big)' + \underset{\substack{\uparrow\\ \text{voltages/}\\ \text{currents}}}{b\big(x(t),t\big)} = 0$$

- given input quantities

$$q_{k+1}^{[n-1]} \approx h^k \frac{d^k}{dt^k} q\big(x(t),t\big)$$

- define

$$\left[\begin{array}{c} q(X_n,t_c) \\ \hline q^{[n]} \end{array}\right] = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{U} \\ \hline \mathcal{B} & \mathcal{V} \end{array}\right] \cdot \left[\begin{array}{c} h\,Q_n' \\ \hline q^{[n-1]} \end{array}\right]$$

  such that

$$A\,Q_n' + b(X_n,t_c) = 0$$

- solve for the stages X_n

GLMs for index-2 DAEs

$$A\ \big(\underbrace{q\big(x(t),t\big)}\big)' + b\big(x(t),t\big) = 0$$

↑ singular

↑ charges/ fluxes

↑ voltages/ currents

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  such that
  $$A\,\mathcal{Q}_n' + b(X_n,t_c) = 0$$
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Remark

- *charge conservation is guaranteed*
- *only charges / fluxes are passed on from step to step*

GLMs for index-2 DAEs

$$A\ \big(\underbrace{q\big(x(t),t\big)}\big)' + b\big(x(t),t\big) = 0$$

↑ singular ↑ charges/ fluxes ↑ voltages/ currents

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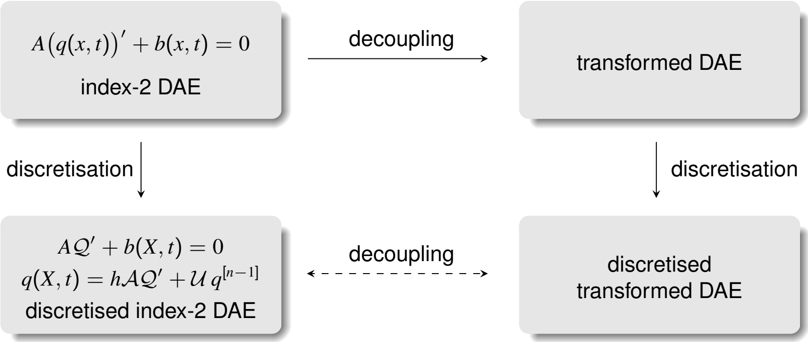
Remark

- *charge conservation is guaranteed*
- *only charges / fluxes are passed on from step to step*
- *. . . this scheme is difficult to analyse . . .*

GLMs for index-2 DAEs (cont.)



$$A\big(q(x,t)\big)' + b(x,t) = 0$$

index-2 DAE

decoupling

transformed DAE

discretisation

discretisation

$$AQ' + b(X,t) = 0$$

$$q(X,t) = h\mathcal{A}Q' + \mathcal{U}\, q^{[n-1]}$$

discretised index-2 DAE

decoupling

discretised transformed DAE

Decoupling index-2 equations

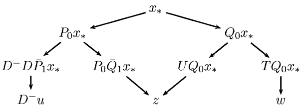
- split solution into characteristic parts
  $$x_*(t) = D^- u(t) + z(t) + w(t)$$

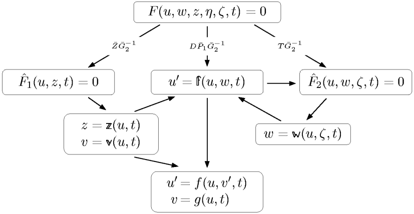
$$\begin{tikzcd}
 & & & x_* \arrow[dll] \arrow[drr] & & & \\
 & P_0 x_* \arrow[dl] \arrow[dr] & & & & Q_0 x_* \arrow[dl] \arrow[dr] & \\
D^- D \bar{P}_1 x_* \arrow[d] & & P_0 \bar{Q}_1 x_* \arrow[dr] & & U Q_0 x_* \arrow[dl] & & T Q_0 x_* \arrow[d] \\
D^- u & & & z & & & w
\end{tikzcd}$$

Decoupling index-2 equations

- split solution into characteristic parts
  $$x_*(t) = D^- u(t) + z(t) + w(t)$$

- split equations similarly
  $$u' = f(u, v', t) \qquad z = \mathbb{z}(u, t)$$
  $$v = g(u, t) \qquad w = \mathbb{w}(u, v', t)$$





Details: *Accessible criteria for the local existence and uniqueness of DAE solutions*, V. (2004), $N_0 \cap S_0$ independent of x

Decoupling index-2 equations

- split solution into characteristic parts
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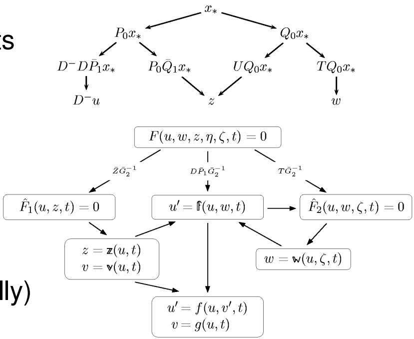
- $I - f_{v'} g_u$ remains non-singular (locally)

$$\boxed{\text{Implicit Index-1 System}}$$



$$\begin{tikzcd}
 & & & x_* \arrow[dll] \arrow[drr] & & & \\
 & P_0x_* \arrow[dl] \arrow[dr] & & & & Q_0x_* \arrow[dl] \arrow[dr] & \\
D^-D\bar{P}_1x_* \arrow[d] & & P_0\bar{Q}_1x_* \arrow[dr] & & UQ_0x_* \arrow[dl] & & TQ_0x_* \arrow[d] \\
D^-u & & & z & & & w
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \boxed{F(u,w,z,\eta,\zeta,t) = 0} \arrow[dll, "\bar{Z}\bar{G}_2^{-1}"'] \arrow[d, "D\bar{P}_1\bar{G}_2^{-1}"] \arrow[drr, "T\bar{G}_2^{-1}"] & & \\
\boxed{\hat{F}_1(u,z,t) = 0} \arrow[dr] & & \boxed{u' = \mathbb{f}(u,w,t)} \arrow[rr] \arrow[dd] & & \boxed{\hat{F}_2(u,w,\zeta,t) = 0} \arrow[dl] \\
 & \boxed{\begin{array}{c} z = \mathbb{z}(u,t) \\ v = \mathbb{v}(u,t) \end{array}} \arrow[ur] \arrow[dr] & & \boxed{w = \mathbb{w}(u,\zeta,t)} \arrow[ul] & \\
 & & \boxed{\begin{array}{c} u' = f(u,v',t) \\ v = g(u,t) \end{array}} & &
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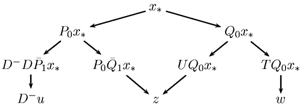
Decoupling index-2 equations



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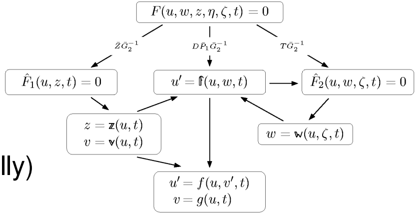

- $I - f_{v'}g_u$ remains non-singular (locally)

$$\boxed{\text{Implicit Index-1 System}}$$

- $$x(t) = D^-(t)u(t) + \mathbb{z}\big(u(t), t\big) + \mathbb{w}\big(u(t), v'(t), t\big)$$

Details: *Accessible criteria for the local existence and uniqueness of DAE solutions*, V. (2004), $N_0 \cap S_0$ independent of x

Decoupling and discretisation

$$\begin{tikzcd}
\begin{array}{c} A\big(q(x,t)\big)' + b(x,t) = 0 \\ \text{index-2 DAE} \end{array} \arrow[r, "\text{decoupling}"] \arrow[d, "\text{discretisation}"'] & \begin{array}{c} u' = f(u,v',t) \\ v = g(u,t) \\ + \text{ constraints} \end{array} \arrow[d, "\text{discretisation}"] \\
\begin{array}{c} AQ' + b(X,t) = 0 \\ q(X,t) = h\mathcal{A}Q' + \mathcal{U}\, q^{[n-1]} \\ \text{discretised index-2 DAE} \end{array} \arrow[r, leftrightarrow, dashed, "\text{decoupling}"] & \begin{array}{c} \text{discretised index-1 DAE} \\ + \text{ discretised constraints} \end{array}
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\end{tikzcd}$$

- If two subspaces associated with the DAE, DN_1 and DS_1 are constant, then this diagram commutes.

GLMs for implicit index-1 DAEs

- $A\big(q(x,t)\big)' + b(x,t) = 0$ is locally equivalent to

$$\begin{aligned} u' &= f(u,v',t) \qquad & z &= \mathbb{z}(u,t) \\ v &= g(u,t) \qquad & w &= \mathbb{w}(u,v',t) \end{aligned}$$

- treat the implicit index-1 system first

GLMs for implicit index-1 DAEs

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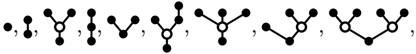
- treat the implicit index-1 system first
- application of $\left[\begin{array}{c|c} \mathcal{A} & \mathcal{U} \\ \hline \mathcal{B} & \mathcal{V} \end{array}\right]$ yields

$$\begin{aligned} U_n &= h\mathcal{A} f(U_n, V_n', t_{nc}) + \mathcal{U}\, u^{[n-1]}, \quad & g(U_n, t_{nc}) &= h\mathcal{A}\, V_n' + \mathcal{U}\, v^{[n-1]} \\ u^{[n]} &= h\mathcal{B} f(U_n, V_n', t_{nc}) + \mathcal{V}\, u^{[n-1]}, \quad & v^{[n]} &= h\mathcal{B}\, V_n' + \mathcal{V}\, v^{[n-1]} \end{aligned}$$

GLMs for index-1 – local error

- find B-series representations of the exact solution

$$\hat{u}^{[1]} = \sum_{\tau \in T} \mathbf{E}(\tau) F(\tau) \frac{\alpha(\tau)\, h^{|\tau|}}{|\tau|!}, \qquad \hat{v}^{[1]} = \sum_{\sigma \in T} \mathbf{E}(\sigma) G(\sigma) \frac{\alpha(\sigma)\, h^{|\sigma|}}{|\sigma|!},$$

$T = \Big\{ \emptyset,$  $, \dots \Big\}$

- find a similar representations for the numerical result

$$u^{[1]} = \sum_{\tau \in T_u,} \mathbf{u}(\tau) F(\tau) \frac{\alpha(\tau)\, h^{|\tau|}}{|\tau|!}, \qquad v^{[1]} = \sum_{\sigma \in T_v,} \mathbf{v}(\sigma) G(\sigma) \frac{\alpha(\sigma)\, h^{|\sigma|}}{|\sigma|!},$$

- order conditions result by comparing coefficients

$$\mathbf{u}(\tau) \overset{?}{=} \mathbf{E}(\tau), \qquad \mathbf{v}(\sigma) \overset{?}{=} \mathbf{E}(\sigma)$$

GLMs for index-1 – local error (cont.)

Theorem

Let $M = \left[\begin{array}{c|ccc} \mathcal{A} & u_1 & \cdots & u_r \\ \hline \mathcal{B} & v_1 & \cdots & v_r \end{array}\right]$ *be in Nordsieck,* $u_i = 0$, $v_i = 0$ *for* $i > r$.

The local error has order $p \in \{1, 2, 3\}$ *if and only if*

order 1

	$\mathcal{B}e + v_2$	$=$	$\begin{pmatrix}1\\1\\0\\0\\\vdots\end{pmatrix}$

order 2

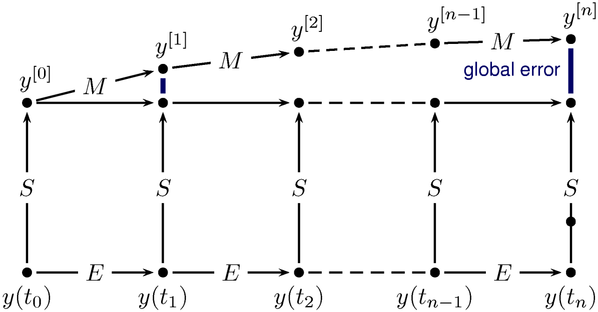
	$\mathcal{B}c + v_3$	$= \frac{1}{2}$	$\begin{pmatrix}1\\2\\2\\0\\\vdots\end{pmatrix}$
	$\mathcal{B}\mathcal{A}^{-1}(c^2 - 2u_3) + 2v_3$	$=$	$\begin{pmatrix}1\\2\\2\\0\\\vdots\end{pmatrix}$

order 3

	$\mathcal{B}c^2 + 2v_4$	$= \frac{1}{3}$
	$\mathcal{B}(\mathcal{A}c + u_3) + v_4$	$= \frac{1}{6}$
	$\mathcal{B}c\mathcal{A}^{-1}(c^2 - 2u_3) + 4v_4$	$= \frac{2}{3}$
	$\mathcal{B}(\mathcal{A}^{-1}(c^2 - 2u_3))^2 + 8v_4$	$= \frac{4}{3}$
	$\mathcal{B}\mathcal{A}^{-1}(c^3 - 6u_4) + 6v_4$	$= 1$
	$\mathcal{B}\mathcal{A}^{-1}(c\mathcal{A}c + cu_3 - 3u_4) + 3v_4$	$= \frac{1}{2}$

$\begin{pmatrix}1\\3\\6\\6\\\vdots\end{pmatrix}$

GLMs for index-1 – global error



- $\epsilon_n = \hat{y}(t_n) - y^{[n]}$ is the global error at $t = t_n$

- **Aim:** find a recursion for the global error $\epsilon_n \mapsto \epsilon_{n+1}$

GLMs for index-1 – global error (cont.)

Lemma

Let $M = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{U} \\ \hline \mathcal{B} & \mathcal{V} \end{array}\right]$ *be a GLM in Nordsieck form with nonsingular* $\mathcal{A}$ *and local error* $\tau_{n+1} = \mathcal{O}(h^{p+1})$.

Then there are suitable projector functions S_n *and* P_n *such that the global error satisfies*

$$\epsilon_{n+1} = (\mathcal{V} - \mathcal{B}\mathcal{A}^{-1}\mathcal{U})\,\mathsf{S}_n\,\epsilon_n + \mathcal{V}\,\mathsf{P}_n\,\epsilon_n + \tau_{n+1} + \mathcal{O}(h^{p+1}).$$

GLMs for index-1 – global error (cont.)

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Then there are suitable projector functions S_n *and* P_n *such that the global error satisfies*

$$\epsilon_{n+1} = (\mathcal{V} - \mathcal{B}\mathcal{A}^{-1}\mathcal{U})\,\mathsf{S}_n\,\epsilon_n + \mathcal{V}\,\mathsf{P}_n\,\epsilon_n + \tau_{n+1} + \mathcal{O}(h^{p+1}).$$

Recall

- $\mathcal{M}(z) = \mathcal{V} + z\,\mathcal{B}\,(I - z\mathcal{A})^{-1}\,\mathcal{U}$ is the method's stability matrix
    - $\mathcal{M}(0) \;= \mathcal{V}$ $\Rightarrow$ *stability at zero*
    - $\mathcal{M}(\infty) = \mathcal{V} - \mathcal{B}\mathcal{A}^{-1}\mathcal{U} = \mathcal{M}_\infty$ $\Rightarrow$ *stability at infinity*

GLMs for index-1 – global error (cont.)

Theorem

Consider nonlinear implicit index-1 DAEs

$$u' = f(u, v'), \qquad v = g(u)$$

and a General Linear Method $M = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{U} \\ \hline \mathcal{B} & \mathcal{V} \end{array}\right]$ *such that*

- $\mathcal{A}$ *is nonsingular*
- *the local error is* $\mathcal{O}(h^{p+1})$
- $\mathcal{M}_\infty = \mathcal{V} - \mathcal{B}\mathcal{A}^{-1}\mathcal{U}$ *and* $\mathcal{V}$ *are power bounded.*

Then the method is convergent with order p*.*

Convergence for index-2 DAEs

$$\begin{tikzcd}
\begin{array}{c} A\big(q(x,t)\big)' + b(x,t) = 0 \\ \text{index-2 DAE} \end{array} \arrow[r, "\text{decoupling}"] \arrow[d, "\text{discretisation}"'] & \begin{array}{c} u' = f(u, v', t) \\ v = g(u,t) \\ + \text{ constraints} \end{array} \arrow[d, "\text{discretisation}"] \\
\begin{array}{c} AQ' + b(X,t) = 0 \\ q(X,t) = hAQ' + \mathcal{U}\, q^{[n-1]} \\ \text{discretised index-2 DAE} \end{array} \arrow[r, leftrightarrow, dashed, "\text{decoupling}"] & \begin{array}{c} \text{discretised index-1 DAE} \\ + \text{ discretised constraints} \end{array}
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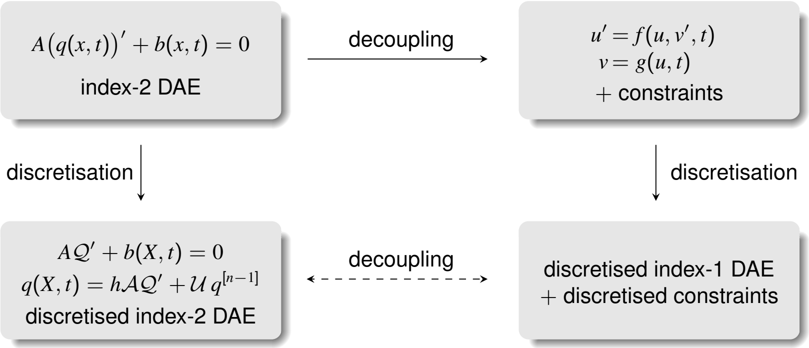
Convergence for index-2 DAEs



$A\big(q(x,t)\big)' + b(x,t) = 0$

index-2 DAE

decoupling

$u' = f(u, v', t)$

$v = g(u,t)$

$+$ constraints

discretisation

discretisation

$AQ' + b(X,t) = 0$

$q(X,t) = hAQ' + \mathcal{U}\, q^{[n-1]}$

discretised index-2 DAE

decoupling

discretised index-1 DAE

$+$ discretised constraints

$$x_n \underset{\uparrow}{=} X_{ns} = D^-(t_{ns})U_{ns} \quad + \mathbb{z}(U_{ns}, t_{ns}) \quad + \mathbb{w}(U_{ns}, V'_{ns}, t_{ns})$$

stiff accuracy → $= D^-(t_{ns})u(t_{ns}) + \mathbb{z}\big(u(t_{ns}), t_{ns}\big) + \mathbb{w}\big(u(t_{ns}), v'(t_{ns}), t_{ns}\big) + \mathcal{O}\big(h^{\min(p-1,q)}\big)$

Convergence for index-2 DAEs (cont.)

Theorem

Consider index-1 DAEs of the form $A\big(q(x,t)\big)' + b(x,t) = 0$ *and a General Linear Method* $M = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{U} \\ \hline \mathcal{B} & \mathcal{V} \end{array}\right]$ *such that*

- $\mathcal{A}$ *is nonsingular*
- *the local error is* $\mathcal{O}(h^{p+1})$ *for implicit index-1 DAEs*
- $\mathcal{M}_\infty$ *and* $\mathcal{V}$ *are power bounded*
- M *is stiffly accurate*
- *the stage order is* q*.*

Then the method is convergent with order $\min(p-1,q)$*.*

Convergence for index-2 DAEs (cont.)

Theorem

Consider index-1 DAEs of the form $A\big(q(x,t)\big)' + b(x,t) = 0$ *and a General Linear Method* $M = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{U} \\ \hline \mathcal{B} & \mathcal{V} \end{array}\right]$ *such that*

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- $\mathcal{M}_\infty$ *and* $\mathcal{V}$ *are power bounded*
- M *is stiffly accurate*
- *the stage order is* q*.*

Then the method is convergent with order $\min(p-1,q)$*.*

If $\mathcal{M}$ *is nilpotent with* $\mathcal{M}^k = 0$*, then, after* $k+1$ *steps, the order of convergence is* $\min(p,q)$*.*

Summary

- we investigated DAEs with index 2 (eg. modified nodal analysis)

$$A\big(q(x,t)\big)' + b(x,t) = 0$$

- GLMs are applied directly to this formulation

$$A\mathcal{Q}' + b(X,t) = 0 \qquad q(X,t) = h\mathcal{A}\mathcal{Q}' + \mathcal{U}\,q^{[n-1]}$$

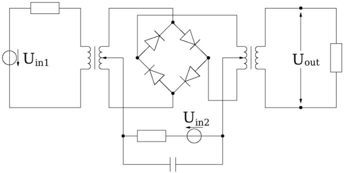
- for *analysing* this scheme we used
  - decoupling procedure, implicit index-1 system
  - trees, B-series and global error recursions

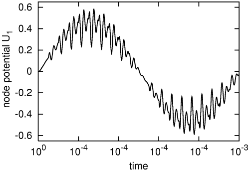
- If the method is
  - stiffly accurate, stable at zero and at infinity, $\mathcal{M}_\infty^k = 0$,

  then the numerical solution converges with order $\min(p,q)$ *for the original formulation*.

Contents

Ringmodulator





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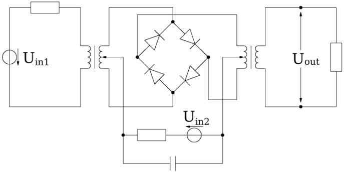

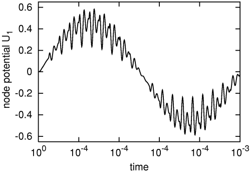

$$M = \left[\begin{array}{cc|cc} \frac{2\lambda-1}{2(\lambda-1)} & 0 & 1 & \frac{2\lambda-1}{2(\lambda-1)} \\ \frac{1-\lambda}{2} & \lambda & 1 & \frac{1-\lambda}{2} \\ \hline \frac{1-\lambda}{2} & \lambda & 1 & \frac{1-\lambda}{2} \\ 0 & 1 & 0 & 0 \end{array}\right]$$

- $\mathcal{A}$ diagonally implicit, $\lambda \in (0, \frac{1}{2})$
- $p = q = 2$
- $\mathcal{V} = \begin{bmatrix} 1 & \frac{1-\lambda}{2} \\ 0 & 0 \end{bmatrix}$, $\mathcal{M}_\infty = \begin{bmatrix} 0 & 0 \\ \frac{\lambda}{1-2\lambda} & 0 \end{bmatrix}$
- stiff accuracy

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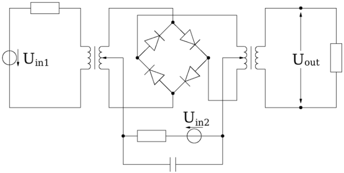

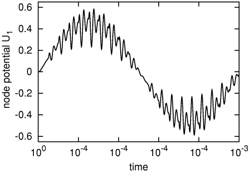

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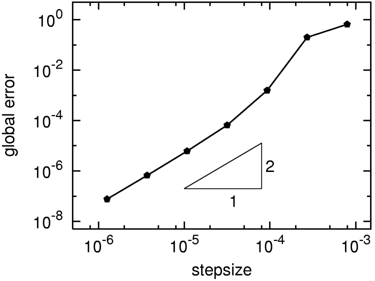

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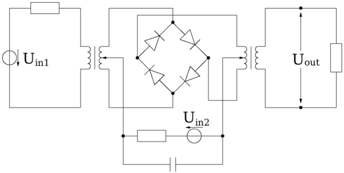

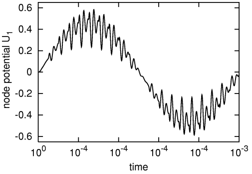

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