CONTACT 3-MANIFOLDS, HOLOMORPHIC CURVES AND INTERSECTION THEORY

EXERCISE SHEET 1

August 27, 2013

Part 1: Lefschetz fibrations

(1) Assume $\pi : M \to \Sigma$ is a 4-dimensional Lefschetz fibration over a surface Σ and $\pi^{-1}(z)$ is a singular fibre consisting of two closed embedded surfaces S_1 and S_2 with a single intersection. Show that their homological intersection numbers then satisfy

$$[S_1] \cdot [S_1] = [S_2] \cdot [S_2] = -1.$$

(2) Show that if $\pi : M \to \Sigma$ is a 4-dimensional Lefschetz fibration and $p \in M$ is a regular point, then the blowup $\widehat{M}$ of M at p naturally inherits a Lefschetz fibration that has one more critical point than π, such that the resulting exceptional sphere is contained in a singular fibre.

Part 2: Holomorphic curves

(3) (a) Show that for a generic compatible almost complex structure J on a closed symplectic 4-manifold (M, ω), if $A \in H_2(M)$ satisfies $c_1(A) = 1$, then the moduli space $\mathcal{M}_0^A(M, J)$ of (equivalence classes up to parametrization of) J-holomorphic spheres homologous to A is *finite*.

(b) If you enjoyed part (a), now prove that if $c_1(A) = 2$ in the above setting, then every nodal curve in $\overline{\mathcal{M}}_0^A(M, J)$ consists of two simple J-holomorphic spheres v_+ and v_- with $c_1([v_\pm]) = 1$.

(4) (a) Consider the intersecting holomorphic maps $u, v : \mathbb{C} \to \mathbb{C}^2$ defined by

$$u(z) = (z^3, z^5), \qquad v(z) = (z^4, z^6).$$

Show that u admits a C^∞-small perturbation to a holomorphic function u_ϵ such that u_ϵ and v have exactly 18 intersections in a neighbourhood of the origin, all transverse.

(b) Try to convince yourself that the above count of 18 intersections holds after *any* generic C^∞-small perturbation of u and/or v.

(c) Show that for any neighbourhood $\mathcal{U} \subset \mathbb{C}$ of 0, the map u admits a C^∞-small perturbation to a holomorphic *immersion* u_ϵ such that

$$\frac{1}{2} \#\{(z, \zeta) \in \mathcal{U} \times \mathcal{U} \mid u_\epsilon(z) = u_\epsilon(\zeta),\ z \neq \zeta\} = 10.$$

(d) If you're especially ambitious, now try to convince yourself that for *any* perturbation as in part (c) making all double points of u_ϵ transverse, the count of double points is the same.

(5) Suppose $L \to \Sigma$ is a complex line bundle over a closed Riemann surface (Σ, j), and $V \subset \Gamma(L)$ is a vector space of sections that satisfy a real-linear Cauchy-Riemann type equation, so in particular the similarity principle holds for sections $\eta \in V$. Prove $\dim_{\mathbb{R}} V \leq 2 + 2c_1(L)$.