CONTACT 3-MANIFOLDS, HOLOMORPHIC CURVES AND INTERSECTION THEORY

EXERCISE SHEET 3

August 29, 2013

(1) In Lecture 3, we saw the theorem of Hofer-Wysocki-Zehnder that for any Reeb orbit $\gamma : S^1 \to M$ in a contact 3-manifold $(M, \xi = \ker \alpha)$, with asymptotic operator $\mathbf{A}_\gamma$ and trivialisation τ of $\gamma^*\xi \to S^1$, the function

$$\sigma(\mathbf{A}_\gamma) \to \mathbb{Z} : \lambda \mapsto \text{wind}^\tau(\lambda) := \text{wind}^\tau(f) \text{ for any nontrivial } f \in \ker(\mathbf{A}_\gamma - \lambda)$$

is well defined, monotone increasing, and attains every value in $\mathbb{Z}$ exactly twice (counting multiplicity of eigenvalues).

(a) Verify that the above theorem holds for the L^2-symmetric operator

$$\mathbf{A} := -J_0 \frac{d}{dt} - c : C^\infty(S^1, \mathbb{R}^2) \to C^\infty(S^1, \mathbb{R}^2),$$

where J_0 denotes the standard complex structure on $\mathbb{R}^2 = \mathbb{C}$ and $c \in \mathbb{R}$ is any constant. (The general case can be derived from this using a deformation argument.)

(b) If $\gamma(t) = \gamma_0(kt)$ for another Reeb orbit $\gamma_0 : S^1 \to M$, then the k-fold cover of each eigenfunction of $\mathbf{A}_{\gamma_0}$ is an eigenfunction of $\mathbf{A}_\gamma$. Assuming τ is the pullback under $S^1 \to S^1 : t \mapsto kt$ of a trivialisation of $\gamma_0^*\xi \to S^1$, show that a nontrivial eigenfunction f of $\mathbf{A}_\gamma$ is a k-fold cover if and only if $\text{wind}^\tau(f)$ is divisible by k.

(c) Assume γ_0 is an embedded orbit that is k-fold covered by γ, and τ is defined by pulling back a trivialisation of $\gamma_0^*\xi \to S^1$. Show that for any nontrivial eigenfunction f of $\mathbf{A}_\gamma$,

$$\text{cov}(f) := \max\{k \in \mathbb{N} \mid f \text{ is a } k\text{-fold cover}\} = \gcd(k, \text{wind}^\tau(f)).$$

(d) Show that if γ is a Reeb orbit that has even Conley-Zehnder index, then so does every multiple cover γ^k of γ.

(2) Assume $\gamma : S^1 \to M$ is a nondegenerate Reeb orbit in a contact 3-manifold $(M, \xi = \ker \alpha)$, with covering multiplicity

$$\text{cov}(\gamma) = \max\left\{k \in \mathbb{N} \mid \gamma(t + 1/k) = \gamma(t) \text{ for all } t \in S^1\right\}.$$

Given $J \in \mathcal{J}(\alpha)$, let $u_\gamma : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ denote the associated J-holomorphic orbit cylinder.

(a) Show that $c_N(u_\gamma) = -p(\gamma)$, where $p(\gamma) \in \{0, 1\}$ is the parity of the Conley-Zehnder index of γ.

(b) Show that $u_\gamma * u_\gamma = -\text{cov}(\gamma) \cdot p(\gamma)$.

(c) Deduce from part (b) that if u^k denotes a k-fold cover of a given asymptotically cylindrical J-holomorphic curve u, it is *not* generally true that $u^k * v^\ell = k\ell(u * v)$.
Remark: One can show however that in general,

$$u^k * v^\ell \geq k\ell(u * v).$$

(d) (*)Use the adjunction formula to show the following: if γ is a multiple cover of a Reeb orbit with even Conley-Zehnder index, and J' is an arbitrary almost complex structure on $\mathbb{R} \times M$ that is compatible with $d(e^s\alpha)$ and belongs to $\mathcal{J}(\alpha)$ outside a compact subset, then there is no simple J'-holomorphic curve homotopic to u_γ through asymptotically cylindrical maps.