Solution Midterm

2. Let X be a topological space, $\gamma : S^1 \to X$ a continuous loop and $D^2 \subset \mathbb{R}^2$ the unit disk. Define $X' := X \sqcup D^2/\sim$ where $z \sim \gamma(z)$.

   (a) Assume that X is Hausdorff and compact, then X' is Hausdorff and compact.

   - Denote $\pi : X \sqcup D^2 \to X'$ the quotient map. By definition of the topology of X' the quotient map π is continuous. Since D^2 is compact and X is compact by assumption, the space X' is the image of a compact space under a continuous map and thus compact.
   - First we claim that π is a closed map. Proof: We need to show that for any closed $A \subset X \sqcup D^2$ the space $\pi(A)$ is closed. By definition the space $\pi(A)$ is closed iff $\pi^{-1}(\pi(A))$ is closed. To show that $\pi^{-1}(\pi(A))$ is closed we distinguish two subcases: case $A \subset X$ or case $A \subset D^2$. For these two cases we have

$$\begin{aligned} &\text{if } A \subset X \text{ then } \pi^{-1}(\pi(A)) = A \sqcup \gamma^{-1}(\gamma(S^1) \cap A) \\ &\text{if } A \subset D^2 \text{ then } \pi^{-1}(\pi(A)) = \gamma(A \cap S^1) \sqcup A\,. \end{aligned}$$

Since S^1 is compact and γ is continuous $\gamma(S^1)$ is compact. Because X is Hausdorff, $\gamma(S^1)$ is also closed. This shows that $\gamma(S^1)\cap A$ is closed and since again γ is continuous $\gamma^{-1}(\gamma(S^1)\cap A)$ is closed. This shows that $\pi^{-1}(\pi(A))$ is closed in the first case. For the second case we argue as follows: $S^1 \cap A$ is closed and since S^1 is compact also compact. This shows that $\gamma(S^1 \cap A)$ is compact and because X is Hausdorff $\gamma(S^1 \cap A)$ is also closed. This shows that $\pi^{-1}(\pi(A))$ is closed in the second case. To see that $\pi^{-1}(\pi(A))$ is closed for a general A note that any closed subspace in $X \sqcup D^2$ is the union of two closed subspaces of the two considered cases.
Having seen that π is closed, we claim that points in X' are closed. Proof: Given a point $[x] \in X'$ and pick any $x \in \pi^{-1}([x])$. Since $X \sqcup D^2$ is Hausdorff, the point set $\{x\}$ is closed. Hence $\{[x]\} = \pi(\{x\})$ is closed.
Having proven the claims, we show that X' is Hausdorff. Proof: Given two points $[x], [y] \in X'$ such that $[x] \neq [y]$. Since points are closed and π is continuous, the spaces $\pi^{-1}([x])$ and $\pi^{-1}([y])$ are closed and disjoint. Since $X \sqcup D^2$ is Hausdorff and compact, it is also normal. Hence we find open and disjoint subsets $U_x, U_y \subset X \sqcup D^2$ such that $\pi^{-1}([x]) \subset U_x$ and $\pi^{-1}([y]) \subset U_y$. Now define

$$\tilde{U}_x := X' \setminus \pi\big((X \sqcup D^2) \setminus U_x\big) \qquad \tilde{U}_y := X' \setminus \pi\big((X \sqcup D^2) \setminus U_y\big)\,.$$

Note that as U_x and U_y are open, their complements A_x and A_y are closed and because π is closed, the spaces $\pi(A_x)$ and $\pi(A_x)$ are closed. This shows that $\tilde{U}_x$ and $\tilde{U}_y$ are open. Before we show that they are disjoint, we claim that

$$\pi^{-1}(\tilde{U}_x) \subset U_x, \qquad \pi^{-1}(\tilde{U}_y) \subset U_y\,.$$

Proof: Take $\tilde{x} \in \pi^{-1}(\tilde{U}_x)$ and assume by contradiction that $\tilde{x} \notin U_x$ or equivalently $\tilde{x} \in (X \sqcup D^2) \setminus U_x$, but then $\pi(\tilde{x})$ must lie in the complement of $\tilde{U}_x$ by definition in contradiction to the choice of $\tilde{x} \in \pi^{-1}(\tilde{U}_x)$. Similarly we show that $\pi^{-1}(\tilde{U}_y) \subset U_y$.
Now we show that $\tilde{U}_x \cap \tilde{U}_y = \emptyset$. Proof: Assume by contradiction that $\tilde{U}_x \cap \tilde{U}_y \neq \emptyset$. Since π is surjective we would have

$$\emptyset \neq \pi^{-1}(\tilde{U}_x \cap \tilde{U}_y) = \pi^{-1}(\tilde{U}_x) \cap \pi^{-1}(\tilde{U}_y) \subset U_x \cap U_y = \emptyset.$$

This is obviously a contradiction. To summarize: we have constructed two disjoint open subsets $\tilde{U}_x$ and $\tilde{U}_y$. It remains to see that $[x] \in \tilde{U}_x$ and $[y] \in \tilde{U}_y$. Proof: Let $x \in \pi^{-1}([x])$ be an element. By construction $x \in U_x$, or equivalently $x \notin A_x$. Thus $[x] = \pi(x) \notin A_x$. Thus $[x] \in \tilde{U}_x$. Similarly we show that $[y] \in \tilde{U}_y$.