Notes on nets and convergence in topology

Nets generalize the notion of sequences so that certain familiar results relating continuity and compactness to sequences in metric spaces can be proved in arbitrary topological spaces. Such a generalization is necessary because, unless one imposes extra conditions on a topological space such as the countability axioms, constructing sequences or subsequences is not always possible the way one would like. The solution is to expand our notion of a "sequence" x_n to something for which the index n need not be a natural number, but can instead take values in a (possibly uncountable) partially ordered set.

1 Nets and sequences

Definition 1.1. A **directed set** (*gerichtete Menge*) $(I, \prec)$ consists of a set I with a partial order[1] $\prec$ such that for every pair $\alpha, \beta \in I$, there exists an element $\gamma \in I$ with $\gamma \succ \alpha$ and $\gamma \succ \beta$.

Example 1.2. The natural numbers $\mathbb{N} = \{1, 2, 3, \dots, \}$ with the relation $\le$ define a directed set $(I, \prec) := (\mathbb{N}, \le)$.

Example 1.3. If X is a topological space and $x \in X$, one can define a directed set $(I, \prec)$ where I is the set of all neighborhoods of x in X, and $\mathcal{U} \prec \mathcal{V}$ for $\mathcal{U}, \mathcal{V} \in I$ means $\mathcal{V} \subset \mathcal{U}$. This is a directed set because given any pair of neighborhoods $\mathcal{U}, \mathcal{V} \subset X$ of x, the intersection $\mathcal{U} \cap \mathcal{V}$ is also a neighborhood of x and thus defines an element of I with $\mathcal{U} \cap \mathcal{V} \subset \mathcal{U}$ and $\mathcal{U} \cap \mathcal{V} \subset \mathcal{V}$. Note that neither of $\mathcal{U}$ and $\mathcal{V}$ need be contained in the other, so they might not satisfy either $\mathcal{U} \prec \mathcal{V}$ or $\mathcal{V} \prec \mathcal{U}$, hence $\prec$ is only a partial order, not a total order (*Totalordnung*). Moreover, for most of the topological spaces we are likely to consider, I is uncountably infinite.

Most directed sets that arise in these notes will be variations on one of the above examples, though we will see a third type of example in the proof of Theorem 4.1.

Definition 1.4. Given a topological space X, a **net** (*Netz*) $\{x_\alpha\}_{\alpha \in I}$ in X is a function $I \to X : \alpha \mapsto x_\alpha$, where $(I, \prec)$ is a directed set.

Definition 1.5. We say that a net $\{x_\alpha\}_{\alpha \in I}$ in X **converges** to $x \in X$ if for every neighborhood $\mathcal{U} \subset X$ of x, there exists an element $\alpha_0 \in I$ such that $x_\alpha \in \mathcal{U}$ for every $\alpha \succ \alpha_0$.

Convergence of nets is also sometimes referred to in the literature as *Moore-Smith convergence*, see e.g. [Kel75].

Example 1.6. A net $\{x_\alpha\}_{\alpha \in I}$ with $(I, \prec) = (\mathbb{N}, \le)$ is simply a sequence, and convergence of this net to x means the same thing as convergence of the sequence.

Definition 1.7. A net $\{x_\alpha\}_{\alpha \in I}$ has a **cluster point** (also known as **accumulation point**, *Häufungspunkt*) at $x \in X$ if for every neighborhood $\mathcal{U} \subset X$ of x and for every $\alpha_0 \in I$, there exists $\alpha \succ \alpha_0$ with $x_\alpha \in \mathcal{U}$.

Definition 1.8. A net $\{y_\beta\}_{\beta \in J}$ is a **subnet** (*Teilnetz*) of the net $\{x_\alpha\}_{\alpha \in I}$ if $y_\beta = x_{\phi(\beta)}$ for some function $\phi : J \to I$ such that for every $\alpha_0 \in I$, there exists an element $\beta_0 \in J$ for which $\beta \succ \beta_0$ implies $\phi(\beta) \succ \alpha_0$.

Example 1.9. If x_n is a sequence, any subsequence x_{k_n} becomes a subnet $\{y_\beta\}_{\beta \in J}$ of the net $\{x_n\}_{n \in \mathbb{N}}$ by setting $J := \mathbb{N}$ and $\phi : \mathbb{N} \to \mathbb{N} : n \mapsto k_n$. Note that this remains true if we slightly relax our notion of subsequences so that k_n need not be a monotone increasing sequence in $\mathbb{N}$ but satisfies $k_n \to \infty$ as $n \to \infty$. Conversely, any subnet $\{y_\beta\}_{\beta \in J}$ of a sequence $\{x_n\}_{n \in \mathbb{N}}$ with $(J, \prec) = (\mathbb{N}, \le)$ is also a subsequence in this slightly relaxed sense, and can then be reduced to a subsequence in the usual sense by skipping some terms (so that the function $n \mapsto k_n$ becomes strictly increasing). Note however that a subnet of a sequence need

[1] Recall that a binary relation $\prec$ defined on a subset of the set of all pairs of elements in I is called a **partial order** (*Halbordnung* or *Teilordung*) if it satisfies (i) $x \prec x$ for all x, (ii) $x \prec y$ and $y \prec x$ implies $x = y$, and (iii) $x \prec y$ and $y \prec z$ implies $x \prec z$. We write "$x \succ y$" as a synonym for "$y \prec x$".

not be a subsequence in general, e.g. it is possible to define a subnet $\{y_\beta\}_{\beta \in J}$ of a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that J is uncountable, and one can derive concrete examples of such objects from the proof of Theorem 4.1 below.

Remark 1.10. If $\{x_\alpha\}_{\alpha\in I}$ is a net converging to x, then every subnet $\{x_{\phi(\beta)}\}_{\beta\in J}$ also converges to x. This follows directly from Definitions 1.5 and 1.8.

It is not true in arbitrary topological spaces that a point is a cluster point of a sequence if and only if it is the limit of a convergent subsequence. But it is true for nets:

Proposition 1.11. *A point $x \in X$ is a cluster point of a net $\{x_\alpha\}_{\alpha\in I}$ in X if and only if there exists a subnet $\{x_{\phi(\beta)}\}_{\beta\in J}$ that converges to x.*

Proof. If $\{x_{\phi(\beta)}\}_{\beta\in J}$ is a subnet of $\{x_\alpha\}_{\alpha\in I}$ converging to x, then for every neighborhood $\mathcal{U} \subset X$ of x, there exists $\beta_0 \in J$ such that $x_{\phi(\beta)} \in \mathcal{U}$ for every $\beta \succ \beta_0$. Then for any $\alpha_0 \in I$, the definition of a subnet implies that we can find $\beta_1 \in J$ with $\phi(\beta) \succ \alpha_0$ for all $\beta \succ \beta_1$, and since J is a directed set, there exists $\beta_2 \in J$ with $\beta_2 \succ \beta_0$ and $\beta_2 \succ \beta_1$. It follows that for $\alpha := \phi(\beta_2)$, $\alpha \succ \alpha_0$ and $x_\alpha = x_{\phi(\beta_2)} \in \mathcal{U}$, thus x is a cluster point of $\{x_\alpha\}_{\alpha\in I}$.

Conversely, if x is a cluster point of $\{x_\alpha\}_{\alpha\in I}$, we can define a convergent subnet as follows. Define a new directed set

$$J = I \times \{\text{neighborhoods of } x \text{ in } X\},$$

with the partial order $(\alpha,\mathcal{U}) \prec (\beta,\mathcal{V})$ defined to mean both $\alpha \prec \beta$ and $\mathcal{V} \subset \mathcal{U}$. Then for each $(\beta,\mathcal{U}) \in J$, the fact that x is a cluster point implies that we can choose $\phi(\beta,\mathcal{U}) \in I$ to be any $\alpha \in I$ such that $\alpha \succ \beta$ and $x_\alpha \in \mathcal{U}$. This defines a function $\phi : J \to I$ such that for any $\alpha_0 \in I$ and any neighborhood $\mathcal{U}_0 \subset X$ of x, every $(\beta,\mathcal{U}) \in J$ with $(\beta,\mathcal{U}) \succ (\alpha_0,\mathcal{U}_0)$ satisfies $\phi(\beta,\mathcal{U}) \succ \beta \succ \alpha_0$, hence $\{x_{\phi(\beta,\mathcal{U})}\}_{\beta\in J}$ is a subnet of $\{x_\alpha\}_{\alpha\in I}$. Moreover, for any neighborhood $\mathcal{U} \subset X$ of x, we can choose an arbitrary $\alpha_0 \in I$ and observe that

$$(\beta,\mathcal{V}) \succ (\alpha_0,\mathcal{U}) \quad \Rightarrow \quad x_{\phi(\beta,\mathcal{V})} \in \mathcal{V} \subset \mathcal{U},$$

thus $\{x_{\phi(\beta,\mathcal{U})}\}_{(\beta,\mathcal{U})\in J}$ converges to x. □

2 The countability axioms

Statements about nets in a topological space can typically be simplified into statements about sequences only if the space satisfies one or both of the countability axioms.

Definition 2.1. Given a point x in a topological space X, a collection of subsets $\{\mathcal{U}_\alpha \subset X\}_{\alpha\in I}$ is called a **neighborhood base** for x if every $\mathcal{U}_\alpha$ is a neighborhood of x and every neighborhood of x contains $\mathcal{U}_\alpha$ for some $\alpha \in I$.

Definition 2.2. A topological space X is called **first countable** (*erfüllt das erste Abzählbarkeitsaxiom*) if every point in X admits a countable neighborhood base.

Example 2.3. All metric (or pseudometric) spaces (X, d) are first countable, as for each $x \in X$, the balls $B_{1/n}(x) = \{y \in X \mid d(y,x) < 1/n\}$ for $n \in \mathbb{N}$ form a countable neighborhood base.

Lemma 2.4. *If $x \in X$ admits a countable neighborhood base, then it also admits a neighborhood base of the form $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \ldots$ such that*

$$X \supset \mathcal{U}_1 \supset \mathcal{U}_2 \supset \mathcal{U}_3 \supset \ldots \ni x.$$

Proof. Suppose $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \ldots$ is a neighborhood base for x. Set $\mathcal{U}_1 = \mathcal{V}_1$, and then define $\mathcal{U}_n$ recursively for each $n \geq 2$ by

$$\mathcal{U}_n = \mathcal{V}_n \cap \mathcal{U}_{n-1}.$$

This is an intersection of two neighborhoods of x and is thus also a neighborhood of x, which satisfies $\mathcal{U}_n \subset \mathcal{U}_{n-1}$ and $\mathcal{U}_n \subset \mathcal{V}_n$ by construction. Then for any other neighborhood $\mathcal{U} \subset X$ of x, the fact that $\mathcal{V}_1, \mathcal{V}_2, \ldots$ is a neighborhood base means that $\mathcal{U} \supset \mathcal{V}_n$ for some $n \in \mathbb{N}$, in which case we also have $\mathcal{U} \supset \mathcal{U}_n$, proving that $\mathcal{U}_1, \mathcal{U}_2, \ldots$ is also a neighborhood base. □

The first countability axiom becomes important in discussions of nets and sequences due to the following result.

Proposition 2.5. *Suppose $x \in X$ has a countable neighborhood base. Then for every net $\{x_\alpha\}_{\alpha \in I}$ in X with a cluster point at x, there exists a sequence $\alpha_1, \alpha_2, \alpha_3, \ldots \in I$ such that the sequence x_{α_n} converges to x, and if $\{x_\alpha\}_{\alpha \in I}$ itself is a sequence, we can take x_{α_n} to be a subsequence.*

Proof. Take a neighborhood base $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \ldots$ consisting of nested neighborhoods as in Lemma 2.4. Then for each $n \in \mathbb{N}$, the fact that x is a cluster point of $\{x_\alpha\}_{\alpha \in I}$ allows us to choose $\alpha_n \in I$ such that $x_{\alpha_n} \in \mathcal{U}_n$. The sequence x_{α_n} converges to x since every neighborhood $\mathcal{V} \subset X$ of x contains $\mathcal{U}_N$ for some $N \in \mathbb{N}$, implying that $x_n \in \mathcal{U}_n \subset \mathcal{U}_N \subset \mathcal{V}$ for every $n \geq N$. Additionally, if $\{x_\alpha\}_{\alpha \in I}$ is a sequence, meaning the directed set $(I, \prec)$ is $(\mathbb{N}, \leq)$, then we can choose the sequence $\alpha_n \in \mathbb{N}$ without loss of generality to be strictly increasing, so that x_{α_n} is a subsequence. $\square$

Definition 2.6. A topological space X is called **second countable** (*erfüllt das zweite Abzählbarkeitsaxiom*) if it admits a countable base.

Example 2.7. We proved on Problem Set 2 #6 that a metric space is second countable whenever it is also **separable**, meaning it admits a countable dense subset. This is true of most of the metric spaces that we commonly think about, e.g. $\mathbb{R}^n$ with its standard Euclidean metric, and many popular function spaces such as the spaces of C^k-smooth functions for $k \geq 0$ and the L^p spaces in measure theory (for $1 \leq p < \infty$). It is also easy to show that finite products and countable disjoint unions of second countable spaces are always second countable. The condition does not hold however for *all* metric spaces, as one can take for instance the discrete metric on any set X: this will be second countable if and only if X itself is countable. (Note however that such a space is only compact if it is finite, so this example is irrelevant for Theorem 4.5 below.)

Proposition 2.8. *Every second countable space is also first countable.*

Proof. If $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \ldots$ is a countable base for the topology of X, then every neighborhood of each point $x \in X$ contains an open neighborhood which is a union of some (necessarily countable) subcollection of $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \ldots$. At least one set $\mathcal{U}_n$ in this subcollection is a neighborhood of x, thus one can take the collection of all $\mathcal{U}_n$ which contain x as a countable neighborhood base for x. $\square$

Lemma 2.9. *If X is a second countable space, then every open cover of X has a countable subcover, i.e. given any collection $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ of open subsets with $X = \bigcup_{\alpha \in I} \mathcal{U}_\alpha$, there exists a countable subset $\{\alpha_1, \alpha_2, \alpha_3, \ldots\} \subset I$ such that $X = \bigcup_{i=n}^\infty \mathcal{U}_{\alpha_n}$.*

Proof. Assume $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ is an open cover of X and $\mathcal{B}$ is a countable neighborhood base. Then each $\mathcal{U}_\alpha$ is a union of sets in $\mathcal{B}$, and the collection of sets in $\mathcal{B}$ that arise in this way is a countable subcollection $\mathcal{B}' \subset \mathcal{B}$. Let us denote $\mathcal{B}' = \{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \ldots\}$, and observe that since $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ covers X, we also have

$$X = \bigcup_{n=1}^\infty \mathcal{V}_n.$$

We can now choose for each $\mathcal{V}_n \in \mathcal{B}'$ an element $\alpha_n \in I$ such that $\mathcal{V}_n \subset \mathcal{U}_{\alpha_n}$, and $\{\mathcal{U}_{\alpha_n}\}_{n \in \mathbb{N}}$ is then a countable subcover of $\{\mathcal{U}_\alpha\}_{\alpha \in I}$. $\square$

3 Continuity

A map $f : X \to Y$ is called **sequentially continuous** (*folgenstetig*) if for every sequence $x_n \in X$ converging to a point $x \in X$, the sequence $f(x_n) \in Y$ converges to $f(x)$. In metric spaces, a standard theorem states that sequential continuity is equivalent to continuity. In arbitrary topological spaces this is no longer true, but we have the following generalization.

Theorem 3.1. *For any two topological spaces X and Y, a map $f : X \to Y$ is continuous if and only if for every net $\{x_\alpha\}_{\alpha \in I}$ in X converging to a point $x \in X$, the net $\{f(x_\alpha)\}_{\alpha \in I}$ in Y converges to $f(x)$.*

Proof. Assume $f : X \to Y$ is continuous and $\{x_\alpha\}_{\alpha \in I}$ is a net in X converging to $x \in X$. Given a neighborhood $\mathcal{U} \subset Y$ of $f(x)$, its preimage $f^{-1}(\mathcal{U}) \subset X$ is a neighborhood of x, so convergence implies that there exists $\alpha_0 \in I$ such that $x_\alpha \in f^{-1}(\mathcal{U})$ for every $\alpha \succ \alpha_0$. This implies $f(x_\alpha) \in \mathcal{U}$ for every $\alpha \succ \alpha_0$ and thus proves that $\{f(x_\alpha)\}_{\alpha \in I}$ converges to $f(x)$.

Conversely, suppose that $f : X \to Y$ is not continuous, so there exists an open set $\mathcal{U} \subset Y$ for which $f^{-1}(\mathcal{U})$ is not open. The latter means $f^{-1}(\mathcal{U})$ contains a point x for which every neighborhood $\mathcal{V} \subset X$ of x contains a point $x_\mathcal{V} \notin f^{-1}(\mathcal{U})$. Define a directed set $(I, \prec)$ as in Example 1.3, i.e. I is the set of all neighborhoods of x, and for two such neighborhoods $\mathcal{V}$ and $\mathcal{V}'$, we write $\mathcal{V} \prec \mathcal{V}'$ whenever $\mathcal{V}' \subset \mathcal{V}$. The points $x_\mathcal{V}$ chosen above then define a net $\{x_\mathcal{V}\}_{\mathcal{V} \in I}$, which converges to x since for every neighborhood $\mathcal{V} \subset X$ of x,

$$\mathcal{V}' \succ \mathcal{V} \quad \Rightarrow \quad x_{\mathcal{V}'} \in \mathcal{V}' \subset \mathcal{V}.$$

If the corresponding net $\{f(x_\mathcal{V})\}_{\mathcal{V} \in I}$ in Y converges to $f(x)$, then since $\mathcal{U} \subset Y$ is an open neighborhood of $f(x)$, we find $\mathcal{V}_0 \in I$ such that $f(x_\mathcal{V}) \in \mathcal{U}$ for all $\mathcal{V} \succ \mathcal{V}_0$. But this means $x_\mathcal{V} \in f^{-1}(\mathcal{U})$ and is thus a contradiction. $\square$

Since sequences are also nets, Theorem 3.1 has the following immediate consequence:

Corollary 3.2. *For any two topological spaces X and Y, all continuous maps $X \to Y$ are also sequentially continuous.*

The converse of this corollary is false, as shown by the following counterexample borrowed from [Jän05, §6.3].

Example 3.3. Let $X = \{f \in [-1,1]^{[0,1]} \mid f \text{ is continuous}\}$, i.e. X is the space of continuous functions $[0,1] \to [-1,1]$, endowed with the subspace topology as a subset of $[-1,1]^{[0,1]} = \prod_{x \in [0,1]} [-1,1]$, so convergence in X means pointwise convergence. We then define

$$Y = \{ f : [0,1] \to \mathbb{R} \mid f \text{ is continuous} \}$$

with a metrizable topology determined by the so-called L^2*-metric*,

$$d_{L^2}(f,g) = \left(\int_0^1 |f(t) - g(t)|^2 \, dt \right)^{1/2}.$$

Since every continuous function on $[0,1]$ is also square-integrable, we obtain a well-defined map[2]

$$\Phi : X \to Y : f \mapsto f.$$

We claim that Φ is sequentially continuous. Indeed, if $f_n \to f$ in X, then the functions f_n converge pointwise, but they also satisfy the uniform bound $|f| \le 1$, so appealing to a standard result of measure theory (the Lebesgue dominated convergence theorem), we also have $d_{L^2}(f_n, f) \to 0$ as $n \to \infty$. However, Φ is not continuous. Continuity would imply for instance that the preimage of an ϵ-ball around 0 in the L^2-metric for arbitrarily small $\epsilon > 0$ is an open neighborhood of 0 in X. The topology of $X \subset [-1,1]^{[0,1]}$ has a base consisting of sets of the form

$$\mathcal{U} = \{ f \in X \mid f(x_1) \in \mathcal{U}_1, \dots, f(x_N) \in \mathcal{U}_N \}$$

for finite sets $x_1, \dots, x_N \in [0,1]$ and collections of open subsets $\mathcal{U}_1, \dots, \mathcal{U}_N \subset [-1,1]$. In particular, any neighborhood of 0 in X must contain a set of this form, consisting of continuous functions that are only constrained at finitely many points. One can therefore always find such a function f for which $d_{L^2}(f,0)$ is as close to 1 as desired, so Φ cannot map such neighborhoods into arbitrarily small balls in the L^2-metric.

[2]There is no real good reason to require the functions in either X or Y to be continuous, except that this is the simplest way to ensure that all functions we consider are measurable and have finite L^2-norm. Otherwise the continuity of functions f in X or Y plays no significant role in this discussion.

The trouble with the converse of Corollary 3.2 is that in the last paragraph of our proof of Theorem 3.1, we were able to construct a net $\{x_{\mathcal{V}}\}_{\mathcal{V}\in I}$ whose index set is the (usually uncountable) collection of all neighborhoods of a fixed point x, and in general there is no obvious way to convert this net into a sequence. On the other hand, Proposition 2.5 provides such a sequence whenever x has a countable neighborhood base, and the same argument then works: if f is sequentially continuous then $f(x_n)$ must converge to $f(x)$, implying that $f(x_n)$ belongs to $\mathcal{U}$ for large n even though by construction, x_n never belongs to $f^{-1}(\mathcal{U})$. This gives the same contradiction and thus proves:

Corollary 3.4. *For any two topological spaces X and Y such that X is first countable, all sequentially continuous maps $X \to Y$ are continuous.*

4 Compactness

For metric spaces, the Bolzano-Weierstrass theorem gives an equivalence between compactness (in the sense of open covers having finite subcovers) and **sequential compactness** (*Folgenkompaktheit*), the property that every sequence has a convergent subsequence. Once again the notions of nets and convergent subnets provide the proper language for this discussion in general topological spaces, and we can sometimes deduce additional results about sequential compactness with some assistance from the countability axioms. We first prove the natural generalization of the Bolzano-Weierstrass theorem.

Theorem 4.1. *A topological space X is compact if and only if every net in X has a convergent subnet.*

Proof. Suppose X is compact but there exists a net $\{x_\alpha\}_{\alpha\in I}$ in X with no cluster point; recall that by Prop. 1.11, having a cluster point is equivalent to having a convergent subnet. The fact that every $x \in X$ is not a cluster point of $\{x_\alpha\}_{\alpha\in I}$ then means that we can find for each $x \in X$ an open neighborhood $\mathcal{U}_x \subset X$ of x and an index $\alpha_x \in I$ such that $x_\alpha \notin \mathcal{U}_x$ for all $\alpha \succ \alpha_x$. But $\{\mathcal{U}_x\}_{x\in X}$ is then an open cover of X and therefore has a finite subcover, meaning there is a finite subset $x_1, \ldots, x_N \in X$ such that $X = \bigcup_{n=1}^N \mathcal{U}_{x_n}$. Since $(I, \prec)$ is a directed set, there also exists an element $\beta \in I$ such that

$$\beta \succ \alpha_{x_n} \quad \text{for each} \quad n = 1, \ldots, N.$$

Then $x_\beta \notin \mathcal{U}_{x_n}$ for every $n = 1, \ldots, N$, but since the sets $\mathcal{U}_{x_n}$ cover X, this is a contradiction.

Conversely, suppose that every net in X has a cluster point, but that X has a collection $\mathcal{O}$ of open sets that cover X such that no finite subcollection in $\mathcal{O}$ covers X. Define a directed set $(I, \prec)$ where I is the set of all finite subcollections of $\mathcal{O}$, with the ordering relation defined by inclusion, i.e. for $\mathcal{A}, \mathcal{B} \in I$, $\mathcal{A} \prec \mathcal{B}$ means $\mathcal{A} \subset \mathcal{B}$. Note that $(I, \prec)$ is a directed set since for any two $\mathcal{A}, \mathcal{B} \in I$, we have $\mathcal{A} \cup \mathcal{B} \in I$ with $\mathcal{A} \cup \mathcal{B} \supset \mathcal{A}$ and $\mathcal{A} \cup \mathcal{B} \supset \mathcal{B}$. By assumption, none of the unions $\bigcup_{\mathcal{U}\in\mathcal{A}} \mathcal{U}$ for $\mathcal{A} \in I$ cover X, so we can choose a point

$$x_{\mathcal{A}} \in X \setminus \bigcup_{\mathcal{U}\in\mathcal{A}} \mathcal{U} \tag{4.1}$$

for each $\mathcal{A} \in I$, thus defining a net $\{x_{\mathcal{A}}\}_{\mathcal{A}\in I}$. Then $\{x_{\mathcal{A}}\}_{\mathcal{A}\in I}$ has a cluster point $x \in X$. Since the sets in $\mathcal{O}$ cover X, we have $x \in \mathcal{V}$ for some $\mathcal{V} \in \mathcal{O}$, and the collection $\{\mathcal{V}\}$ is an element of I, hence there exists $\mathcal{A} \succ \{\mathcal{V}\}$ such that $x_{\mathcal{A}} \in \mathcal{V}$. But this means $\mathcal{A}$ is a finite subcollection of $\mathcal{O}$ that includes $\mathcal{V}$, thus contradicting (4.1). □

The next two examples show that neither direction of Theorem 4.1 holds in general for sequences, at least not without further conditions on the space X.

Example 4.2 (cf. Problem Set 3 # 1)**.** The space $[0,1]^{\mathbb{R}}$ of arbitrary functions $\mathbb{R} \to [0,1]$ with the topology of pointwise convergence is compact according to Tychonoff's theorem, as it has a natural identification with the infinite product $\prod_{x\in\mathbb{R}}[0,1]$, and $[0,1]$ is compact. But one can define a sequence $f_n \in [0,1]^{\mathbb{R}}$ with no convergent subsequence as follows. For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, let $x_{(n)} \in \{0, \ldots, 9\}$ denote the nth digit to the right of the decimal point in the decimal expansion of x. Then defining $f_n(x) = x_{(n)}/10$ gives a sequence $f_n \in [0,1]^{\mathbb{R}}$, and for every subsequence f_{k_n} there exists a point $x \in \mathbb{R}$ at which $f_{k_n}(x)$ does not converge as $n \to \infty$.

Example 4.3 (cf. Problem Set 3 # 2)**.** In the space $[0,1]^{\mathbb{R}}$ from Example 4.2, consider the subset

$$X = \left\{ f \in [0,1]^{\mathbb{R}} \;\middle|\; f(x) \neq 0 \text{ for at most countably many points } x \in \mathbb{R} \right\},$$

with the subspace topology. For any sequence $f_n \in X$, the set $\bigcup_{n \in \mathbb{N}} \{x \in \mathbb{R} \mid f_n(x) \neq 0\}$ is a countable union of countable sets and is thus also countable; denote its elements by $t_1, t_2, t_3, \ldots$. Since $[0,1]$ is compact, we can recursively define a sequence of subsequences as follows: choose $f_n^{(1)}$ to be any subsequence of f_n such that $f_n^{(1)}(t_1)$ converges, and for each integer $m \geq 2$, choose $f_n^{(m)}$ to be a subsequence of $f_n^{(m-1)}$ such that $f_n^{(m)}(t_m)$ converges—this construction ensures that in fact $f_n^{(m)}(t_j)$ converges for each $j = 1, \ldots, m$. Then the diagonal sequence $f_n^{(n)}$ is a subsequence of f_n that converges at all of the points $t_1, t_2, t_3, \ldots$ and vanishes everywhere else, thus it is a pointwise convergent subsequence.

On the other hand, it is easy to see that X is not compact: since every function in X vanishes somewhere, the collection $\{\mathcal{U}_x \subset X \mid x \in \mathbb{R}\}$ where $\mathcal{U}_x := \{f \in X \mid f(x) = 0\}$ forms an open cover of X, but it has no finite subcover since there exists no finite subset of $\mathbb{R}$ on which every $f \in X$ is guaranteed to vanish somewhere.

As in the previous section, converting Theorem 4.1 into a statement about sequential compactness requires the countability axioms. In one direction, the result follows immediately by combining the theorem with Proposition 2.5: if X is compact, then the theorem guarantees that every sequence x_n has a cluster point $x \in X$, so if x is also known to have a countable neighborhood base, Proposition 2.5 extracts from x_n a subsequence converging to x. We've proved:

Corollary 4.4. *Every compact topological space that is first countable is also sequentially compact.*

For the other direction, we can repeat more or less the same argument as in Theorem 4.1, but using the axiom of second countability to replace nets with sequences.

Theorem 4.5. *If X is a second countable topological space that is sequentially compact, then it is compact.*

Proof. We need to show that every open cover of X has a finite subcover. Since X is second countable, we can first use Lemma 2.9 to reduce the given open cover to a *countable* subcover $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \ldots \subset X$. Now arguing by contradiction, suppose that X is sequentially compact but the sets $\mathcal{U}_1, \ldots, \mathcal{U}_n$ do not cover X for any $n \in \mathbb{N}$, hence there exists a sequence $x_n \in X$ such that

$$x_n \in X \setminus (\mathcal{U}_1 \cup \ldots \cup \mathcal{U}_n) \tag{4.2}$$

for every $n \in \mathbb{N}$. Some subsequence x_{k_n} then converges to a point $x \in X$, which necessarily lies in $\mathcal{U}_N$ for some $N \in \mathbb{N}$. It follows that x_{k_n} also lies in $\mathcal{U}_N$ for all n sufficiently large, but this contradicts (4.2) as soon as $k_n \geq N$. □

5 Epilogue: Products of compact spaces

We can now use Theorem 4.1 to provide a very quick proof of the "finite" case of Tychonoff's theorem.[3]

Theorem 5.1. *If X and Y are compact topological spaces, then so is $X \times Y$.*

Proof. Suppose $\{(x_\alpha, y_\alpha)\}_{\alpha \in I}$ is a net in $X \times Y$. Then the compactness of X implies that the net $\{x_\alpha\}_{\alpha \in I}$ has a convergent subnet $\{x_{\phi(\beta)}\}_{\beta \in J}$, and the compactness of Y implies in turn that the net $\{y_{\phi(\beta)}\}_{\beta \in J}$ in Y has a convergent subnet $\{y_{\phi \circ \psi(\gamma)}\}_{\gamma \in K}$. We conclude that

$$\{(x_{\phi \circ \psi(\gamma)}, y_{\phi \circ \psi(\gamma)}\}_{\gamma \in K}$$

is a convergent subnet of the original net in $X \times Y$. □

If you followed that argument and are familiar with Zorn's lemma (a statement about existence of maximal elements in partially ordered sets which is equivalent to the axiom of choice), then you might now be interested in reading Chernoff's proof [Che92] of Tychonoff's theorem for infinite products. The paper is quite short and readable if you're in the right mood. Zorn's lemma itself is discussed e.g. in [Kel75] and [Jän05], both of which also include alternative proofs of Tychonoff's theorem.

[3] It is also possible to prove this result directly in terms of open covers and finite subcovers, and that argument can be found in many of the standard books on point-set topology, but I find it a bit tedious.

References

[Che92] P. R. Chernoff, *A simple proof of Tychonoff's theorem via nets*, Amer. Math. Monthly **99** (1992), no. 10, 932–934.

[Jän05] K. Jänich, *Topologie*, 8th ed., Springer-Verlag, Berlin, 2005 (German).

[Kel75] J. L. Kelley, *General topology*, Graduate Texts in Mathematics, vol. 27, Springer-Verlag, New York, 1975. Reprint of the 1955 edition [Van Nostrand, Toronto, Ont.]