PROBLEM SET 7
Due: 7.06.2017

Instructions

Problems marked with $(*)$ will be graded. Solutions may be written up in German or English and should be handed in before the Übung on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Wednesday lecture.

Special note: *You may continue to use the fact that $\pi_1(S^1) \cong \mathbb{Z}$ without proof. (We'll prove it next week.)*

Problems

1. Recall that the wedge sum of two pointed spaces (X, x) and (Y, y) is defined as $X \vee Y = (X \sqcup Y)/\sim$ where the equivalence relation identifies the two base points x and y. It is commonly said that whenever X and Y are both path-connected and are otherwise "reasonable" spaces, the formula

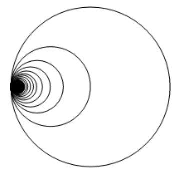
$$\pi_1(X \vee Y) \cong \pi_1(X) * \pi_1(Y) \tag{1}$$

holds. We've seen for instance that this is true when X and Y are both circles. The goal of this problem is to understand slightly better what "reasonable" means in this context, and why such a condition is needed.

(a) Show by a direct argument (i.e. without trying to use Seifert-van Kampen) that if X and Y are both Hausdorff and simply connected, then $X \vee Y$ is simply connected.
Hint: Hausdorff implies that $X \setminus \{x\}$ and $Y \setminus \{y\}$ are both open subsets. Consider loops $\gamma : [0,1] \to X \vee Y$ based at $[x] = [y]$ and decompose $[0,1]$ into subintervals in which $\gamma(t)$ stays in either X or Y.

(b) Call a pointed space (X, x) *nice*[1] if X is Hausdorff and x admits an open neighborhood that is simply connected. Show that the formula (1) holds whenever (X, x) and (Y, y) are both nice.

(c) Here is an example of a space that is not "nice" in the sense of part (b): the so-called *Hawaiian earring* can be defined as the subset of $\mathbb{R}^2$ consisting of the union for all $n \in \mathbb{N}$ of the circles of radius $1/n$ centered at $(1/n, 0)$. As usual, we assign to this set the subspace topology induced by the standard topology of $\mathbb{R}^2$. Show that in this space, the point $(0, 0)$ does not have any simply connected open neighborhood.



(d) It is tempting to liken the Hawaiian earring to the infinite wedge sum of circles $X := \bigvee_{n=1}^{\infty} S^1$, defined as above by choosing a base point in each copy of the circle and then identifying all the base points in the infinite disjoint union $\bigsqcup_{n=1}^{\infty} S^1$. Since both X and the Hawaiian earring are unions of infinite collections of circles that all intersect each other at one point, it is not hard to imagine a bijection between them. Show however that such a bijection can never be a homeomorphism; in particular, unlike the Hawaiian earring, X is "nice" for any choice of base point.
Hint: Pay attention to how the topology of X is defined—it is a quotient of a disjoint union.

2. $(*)$ As proved in lecture, the closed orientable surface Σ_g of genus $g \geq 0$ has $\pi_1(\Sigma_g)$ isomorphic to

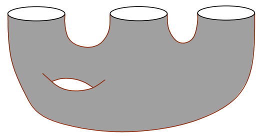
$$G_g := \left\{ x_1, y_1, \ldots, x_g, y_g \;\middle|\; x_1 y_1 x_1^{-1} y_1^{-1} \ldots x_g y_g x_g^{-1} y_g^{-1} = e \right\}.$$

Show that the abelianization (cf. Problem Set 6 #2) of G_g is isomorphic to the additive group $\mathbb{Z}^{2g}$.
Hint: By definition, G_g is a particular quotient of the free group on $2g$ generators. Observe that the abelianization of the latter is exactly the same group as the abelianization of G_g. (Why?)

Remark: $\mathbb{Z}^n$ is isomorphic to $\mathbb{Z}^m$ if and only if $n = m$, so this proves Σ_g and Σ_h are not homeomorphic.

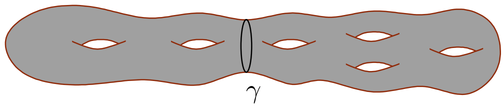
[1] Not a standardized term, I made it up.

3. For integers $g, m \geq 0$, let $\Sigma_{g,m}$ denote the compact surface obtained by cutting m disjoint disk-shaped holes out of the closed orientable surface with genus g. (By this convention, $\Sigma_g = \Sigma_{g,0}$.) The boundary $\partial\Sigma_{g,m}$ is then a disjoint union of m circles, e.g. the case with $g = 1$ and $m = 3$ might look like the picture at the right.



   (a) $(*)$ Show that $\pi_1(\Sigma_{g,1})$ is a free group with $2g$ generators, and if $g \geq 1$, then any simple closed curve parametrizing $\partial\Sigma_{g,1}$ represents a nontrivial element of $\pi_1(\Sigma_{g,1})$.[2]
   Hint: Think of Σ_g as a polygon with some of its edges identified. If you cut a hole in the middle of the polygon, what remains admits a deformation retraction to the edges. Prove it with a picture.

   (b) $(*)$ Assume γ is a simple closed curve separating Σ_g into two pieces homeomorphic to $\Sigma_{h,1}$ and $\Sigma_{k,1}$ for some $h, k \geq 0$. (The picture at the right shows an example with $h = 2$ and $k = 4$.) Show that the image of $[\gamma] \in \pi_1(\Sigma_g)$ under the natural projection to the abelianization of $\pi_1(\Sigma_g)$ is trivial.



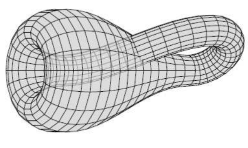
   Hint: What does γ look like in the polygonal picture from part (a)? What is it homotopic to?

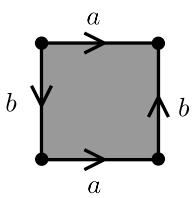
   (c) $(*)$ Show that the curve γ in part (b) nevertheless represents a nontrivial element of $\pi_1(\Sigma_g)$.
   Hint: Break Σ_g into two open subsets overlapping near γ, then see what van Kampen tells you.

   (d) Prove that the generators a and b in the group $\{a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = e\}$ satisfy $ab \neq ba$.
   Advice: A purely algebraic solution to this problem is presumably possible, but you could also just deduce it from part (c).

   (e) Generalize part (a): show that if $m \geq 1$, $\pi_1(\Sigma_{g,m})$ is a free group with $2g + m - 1$ generators.

4. The first of the two pictures at the right shows one of the standard ways of representing the *Klein bottle*[3]as an "immersed" (i.e. smooth but with self-intersections) surface in $\mathbb{R}^3$. As a topological space, the technical definition is





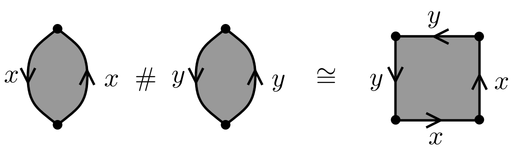
$$\mathbb{K}^2 = [0,1]^2/\sim$$

where $(s, 0) \sim (s, 1)$ and $(0, t) \sim (1, 1 - t)$ for every $s, t \in [0, 1]$. This is represented by the square with pairs of sides identified in the rightmost picture; notice the reversal of arrows, which is why $\mathbb{K}^2 \neq \mathbb{T}^2$!

   (a) Using the same argument by which we computed $\pi_1(\Sigma_g)$ in lecture, show that $\pi_1(\mathbb{K}^2)$ is isomorphic to $G := \{a, b \mid aba^{-1}b = e\}$.

   (b) $(*)$ Consider the subset $\ell = \{[(s,t)] \in \mathbb{K}^2 \mid t = 1/4 \text{ or } t = 3/4\}$ in $\mathbb{K}^2$. Show that ℓ is a simple closed curve which separates $\mathbb{K}^2$ into two pieces, each homeomorphic to the Möbius band $\mathbb{M}^2 := \{(e^{i\theta}, \tau e^{i\theta/2}) \in S^1 \times \mathbb{C} \mid \theta \in [0, 2\pi],\ \tau \in [-1, 1]\}$. Use this decomposition to show via the Seifert-van Kampen theorem that $\pi_1(\mathbb{K}^2)$ is also isomorphic to $G' := \{c, d \mid c^2 = d^2\}$.

   (c) Recall that $\mathbb{RP}^2$ can be constructed by gluing $\mathbb{M}^2$ to a disk $\mathbb{D}^2$, so conversely, $\mathbb{RP}^2 \setminus \mathring{\mathbb{D}}^2 \cong \mathbb{M}^2$. Part (b) implies therefore that $\mathbb{K}^2$ is homoemorphic to the connected sum $\mathbb{RP}^2 \# \mathbb{RP}^2$ (cf. Problem Set 6 #3). Now, viewing $\mathbb{RP}^2$ as a polygon with two (curved) edges that are identified, imitate the argument we carried out for Σ_g in lecture to derive a different presentation for $\mathbb{K}^2$ as shown in the figure below, and deduce that $\pi_1(\mathbb{K}^2)$ is also isomorphic to $G'' := \{x, y \mid x^2y^2 = e\}$.



   (d) For the groups G, G' and G'' above, find explicit isomorphisms of their abelianizations to $\mathbb{Z} \oplus \mathbb{Z}_2$. Then find explicit isomorphisms from each of G, G' and G'' to the others.

[2]Terminology: one says in this case that $\partial\Sigma_{g,1}$ is *homotopically nontrivial* or *essential*, or equivalently, *not nullhomotopic*.
[3]If you think my glass Klein bottle is cool, you can buy your own at `http://www.kleinbottle.com/`.