TOPOLOGY I — HUMBOLDT-UNIVERSITÄT ZU BERLIN
C. WENDL — SUMMER SEMESTER 2018

**PROBLEM SET 7**
**Due: 12.06.2018**

### Instructions

Problems marked with $(*)$ will be graded. Solutions may be written up in German or English and should be handed in before the Übung on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Tuesday lecture.

### Problems

1. Recall that the wedge sum of two pointed spaces $(X, x)$ and $(Y, y)$ is defined as $X \vee Y = (X \amalg Y)/\sim$ where the equivalence relation identifies the two base points $x$ and $y$. It is commonly said that whenever $X$ and $Y$ are both path-connected and are otherwise "reasonable" spaces, the formula

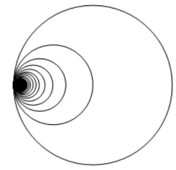
$$\pi_1(X \vee Y) \cong \pi_1(X) * \pi_1(Y) \tag{1}$$

holds. We've seen for instance that this is true when $X$ and $Y$ are both circles. The goal of this problem is to understand slightly better what "reasonable" means in this context, and why such a condition is needed.

(a) Show by a direct argument (i.e. without trying to use Seifert-van Kampen) that if $X$ and $Y$ are both Hausdorff and simply connected, then $X \vee Y$ is simply connected.
*Hint: Hausdorff implies that $X \backslash \{x\}$ and $Y \backslash \{y\}$ are both open subsets. Consider loops $\gamma : [0,1] \to X \vee Y$ based at $[x] = [y]$ and decompose $[0,1]$ into subintervals in which $\gamma(t)$ stays in either $X$ or $Y$.*

(b) Call a pointed space $(X, x)$ *nice*[1] if $x$ has an open neighborhood that admits a deformation retraction to $x$.[2] Show that the formula (1) holds whenever $(X, x)$ and $(Y, y)$ are both nice.

(c) Here is an example of a space that is not "nice" in the sense of part (b): the so-called *Hawaiian earring* can be defined as the subset of $\mathbb{R}^2$ consisting of the union for all $n \in \mathbb{N}$ of the circles of radius $1/n$ centered at $(1/n, 0)$. As usual, we assign to this set the subspace topology induced by the standard topology of $\mathbb{R}^2$. Show that in this space, the point $(0, 0)$ does not have any simply connected open neighborhood.



(d) It is tempting to liken the Hawaiian earring to the infinite wedge sum of circles $X := \bigvee_{n=1}^{\infty} S^1$, defined as above by choosing a base point in each copy of the circle and then identifying all the base points in the infinite disjoint union $\coprod_{n=1}^{\infty} S^1$. Since both $X$ and the Hawaiian earring are unions of infinite collections of circles that all intersect each other at one point, it is not hard to imagine a bijection between them. Show however that such a bijection can never be a homeomorphism; in particular, unlike the Hawaiian earring, $X$ is "nice" for any choice of base point.
*Hint: Pay attention to how the topology of $X$ is defined—it is a quotient of a disjoint union.*

2. $(*)$ As proved in lecture, the closed orientable surface $\Sigma_g$ of genus $g \geqslant 0$ has $\pi_1(\Sigma_g)$ isomorphic to

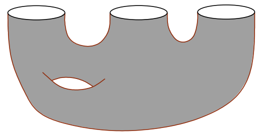
$$G_g := \left\{ x_1, y_1, \ldots, x_g, y_g \mid x_1 y_1 x_1^{-1} y_1^{-1} \ldots x_g y_g x_g^{-1} y_g^{-1} = e \right\}.$$

Show that the abelianization (cf. Problem Set 6 #2) of $G_g$ is isomorphic to the additive group $\mathbb{Z}^{2g}$.
*Hint: By definition, $G_g$ is a particular quotient of the free group on $2g$ generators. Observe that the abelianization of the latter is exactly the same group as the abelianization of $G_g$. (Why?)*

*Remark: $\mathbb{Z}^n$ is isomorphic to $\mathbb{Z}^m$ if and only if $n = m$, so this proves $\Sigma_g$ and $\Sigma_h$ are not homeomorphic.*

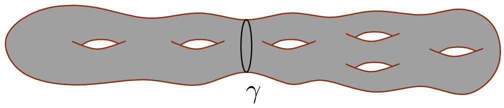
[1] Not a standardized term, I made it up.
[2] This is a revised version of the problem sheet. The definition of "nice" has been strengthened from the original version.

3. For integers $g, m \geqslant 0$, let $\Sigma_{g,m}$ denote the compact surface obtained by cutting $m$ disjoint disk-shaped holes out of the closed orientable surface with genus $g$. (By this convention, $\Sigma_g = \Sigma_{g,0}$.) The boundary $\partial\Sigma_{g,m}$ is then a disjoint union of $m$ circles, e.g. the case with $g = 1$ and $m = 3$ might look like the picture at the right.



   (a) $(*)$ Show that $\pi_1(\Sigma_{g,1})$ is a free group with $2g$ generators, and if $g \geqslant 1$, then any simple closed curve parametrizing $\partial\Sigma_{g,1}$ represents a nontrivial element of $\pi_1(\Sigma_{g,1})$.[3]
   *Hint: Think of $\Sigma_g$ as a polygon with some of its edges identified. If you cut a hole in the middle of the polygon, what remains admits a deformation retraction to the edges. Prove it with a picture.*

   (b) $(*)$ Assume $\gamma$ is a simple closed curve separating $\Sigma_g$ into two pieces homeomorphic to $\Sigma_{h,1}$ and $\Sigma_{k,1}$ for some $h, k \geqslant 0$. (The picture at the right shows an example with $h = 2$ and $k = 4$.) Show that the image of $[\gamma] \in \pi_1(\Sigma_g)$ under the natural projection to the abelianization of $\pi_1(\Sigma_g)$ is trivial.
   *Hint: What does $\gamma$ look like in the polygonal picture from part (a)? What is it homotopic to?*

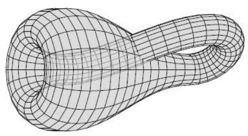


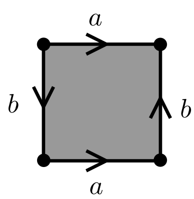

   (c) $(*)$ Show that if $h > 0$ and $k > 0$, then the curve $\gamma$ in part (b) nevertheless represents a nontrivial element of $\pi_1(\Sigma_g)$.[4]
   *Hint: Break $\Sigma_g$ into two open subsets overlapping near $\gamma$, then see what van Kampen tells you.*

   (d) Prove that the generators $a$ and $b$ in the group $\{a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = e\}$ satisfy $ab \neq ba$.
   *Advice: A purely algebraic solution to this problem is presumably possible, but you could also just deduce it from part (c).*

   (e) Generalize part (a): show that if $m \geqslant 1$, $\pi_1(\Sigma_{g,m})$ is a free group with $2g + m - 1$ generators.

4. The first of the two pictures at the right shows one of the standard ways of representing the *Klein bottle*[5] as an "immersed" (i.e. smooth but with self-intersections) surface in $\mathbb{R}^3$. As a topological space, the technical definition is






$$\mathbb{K}^2 = [0,1]^2/\sim$$

where $(s, 0) \sim (s, 1)$ and $(0, t) \sim (1, 1 - t)$ for every $s, t \in [0, 1]$. This is represented by the square with pairs of sides identified in the rightmost picture; notice the reversal of arrows, which is why $\mathbb{K}^2 \neq \mathbb{T}^2$!

   (a) Using the same argument by which we computed $\pi_1(\Sigma_g)$ in lecture, show that $\pi_1(\mathbb{K}^2)$ is isomorphic to $G := \{a, b \mid aba^{-1}b = e\}$.

   (b) $(*)$ Consider the subset $\ell = \{[(s, t)] \in \mathbb{K}^2 \mid t = 1/4 \text{ or } t = 3/4\}$ in $\mathbb{K}^2$. Show that $\ell$ is a simple closed curve which separates $\mathbb{K}^2$ into two pieces, each homeomorphic to the Möbius band $\mathbb{M}^2 := \{(e^{i\theta}, \tau e^{i\theta/2}) \in S^1 \times \mathbb{C} \mid \theta \in [0, 2\pi],\ \tau \in [-1, 1]\}$. Use this decomposition to show via the Seifert-van Kampen theorem that $\pi_1(\mathbb{K}^2)$ is also isomorphic to $G' := \{c, d \mid c^2 = d^2\}$.

   (c) Recall that $\mathbb{RP}^2$ can be constructed by gluing $\mathbb{M}^2$ to a disk $\mathbb{D}^2$, so conversely, $\mathbb{RP}^2 \backslash \mathring{\mathbb{D}}^2 \cong \mathbb{M}^2$. Part (b) implies therefore that $\mathbb{K}^2$ is homoemorphic to the connected sum $\mathbb{RP}^2 \# \mathbb{RP}^2$ (cf. Problem Set 6 #3). Now, viewing $\mathbb{RP}^2$ as a polygon with two (curved) edges that are identified, imitate the argument we carried out for $\Sigma_g$ in lecture to derive a different presentation for $\mathbb{K}^2$ as shown in the figure below, and deduce that $\pi_1(\mathbb{K}^2)$ is also isomorphic to $G'' := \{x, y \mid x^2 y^2 = e\}$.

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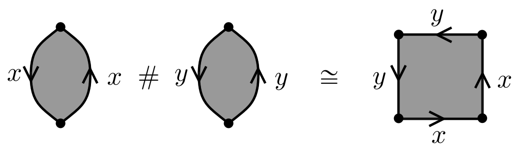
[3]Terminology: one says in this case that $\partial\Sigma_{g,1}$ is *homotopically nontrivial* or *essential*, or equivalently, *not nullhomotopic*.

[4]In this revised version of the problem sheet, the conditions $h > 0$ and $k > 0$ have been added since they are clearly necessary, and you should probably ignore the hint and instead derive part (c) from an algebraic solution to part (d). See the written solutions posted on the website.

[5]If you think my glass Klein bottle is cool, you can buy your own at `http://www.kleinbottle.com/`.




(d) For the groups $G$, $G'$ and $G''$ above, find explicit isomorphisms of their abelianizations to $\mathbb{Z} \oplus \mathbb{Z}_2$. Then find explicit isomorphisms from each of $G$, $G'$ and $G''$ to the others.