TAKE-HOME MIDTERM

Instructions

To receive credit for this assignment, you must hand it in by **Tuesday, June 26** before the Übung. The solutions will be discussed in the Übung on that day.

You are free to use any resources at your disposal and to discuss the problems with your comrades, but **you must write up your solutions alone**. Solutions may be written up in German or English, this is up to you.

There are 100 points in total; a score of 75 points or better will boost your final exam grade according to the formula that was indicated in the course syllabus. Note that the number of points assigned to each part of each problem is usually proportional to its conceptual importance/difficulty.

If a problem asks you to prove something, then unless it says otherwise, a **complete argument** is typically expected, not just a sketch of the idea. Partial credit may sometimes be given for incomplete arguments if you can demonstrate that you have the right idea, but for this it is important to write as clearly as possible. Less complete arguments can sometimes be sufficient, e.g. in cases where you want to show that two spaces are homotopy equivalent and can justify it with a very convincing picture (use your own judgement). You are free to make use of all results we've proved in lectures or problem sets, without reproving them. (When using a result from a problem set, say explicitly which one.)

One more piece of general advice: if you get stuck on one part of a problem, it may often still be possible to move on and do the next part.

You are free to ask for clarification or hints via e-mail or in office hours; of course I reserve the right not to answer such questions.

Problems

1. [10 pts] Prove that a space X is Hausdorff if and only if every convergent net in X has a unique limit. *Remark: We've already proved the analogous statement about sequences, though in one direction it required the assumption that X is first countable. We've also seen an example (see Problem Set 4 #6(g)) of a non-Hausdorff space in which every convergent sequence has a unique limit. In this problem, do not assume X is first countable.*

2. [60 pts total] Recall that for arbitrary sets X and Y, we define the set X^Y to consist of all functions $f : Y \to X$. Here there is no assumption about continuity of functions since we have not even assigned topologies to X and Y. One of the good reasons to use this notation is that there is an obvious bijection

$$Z^{X\times Y} \to (Z^Y)^X$$

sending a function $F : X \times Y \to Z$ to the function $\Phi : X \to Z^Y$ defined by

$$\Phi(x)(y) = F(x, y). \tag{1}$$

The existence of this bijection is sometimes called the *exponential law* for sets. The goal of this problem is to explore to what extent the exponential law carries over to topological spaces and continuous maps.

If X and Y are topological spaces, let us denote by $C(X, Y)$ the space of all continuous maps $X \to Y$, with the compact-open topology, which has a subbase consisting of all sets of the form

$$\mathcal{U}_{K,V} := \{ f \in C(X, Y) \mid f(K) \subset V \}$$

for $K \subset X$ compact and $V \subset Y$ open (see Problem Set 4). Assume Z is also a topological space.

(a) [15 pts] Prove that if $F : X \times Y \to Z$ is continuous, then the correspondence (1) defines a continuous map $\Phi : X \to C(Y, Z)$.

(b) [15 pts] Prove that if Y is locally compact and Hausdorff, then the converse also holds: any continuous map $\Phi : X \to C(Y, Z)$ defines a continuous map $F : X \times Y \to Z$ via (1).

Let's pause for a moment to observe what these two results imply for the case $X := I = [0, 1]$. According to part (a), a homotopy between two maps $Y \to Z$ can always be regarded as a continuous path in $C(Y, Z)$, and part (b) says that the converse is also true if Y is locally compact and Hausdorff, hence two maps $Y \to Z$ are homotopic if and only if they lie in the same path-component of $C(Y, Z)$.[1]

(c) [5 pts] Deduce from part (b) a new proof of the following result from Problem Set 4 #7(d): if X is locally compact and Hausdorff, then the *evaluation map* $\mathrm{ev} : C(X, Y) \times X \to Y : (f, x) \mapsto f(x)$ is continuous.
Hint: This is very easy if you look at it from the right perspective.
Remark: If you were curious to see a counterexample to part (b) in a case where Y is not locally compact, you could now extract one from Problem Set 4 #7(f).

(d) [15 pts] The following cannot be deduced directly from part (b), but it is a similar result and requires a similar proof: show that if Y is locally compact and Hausdorff, then

$$C(X, Y) \times C(Y, Z) \to C(X, Z) : (f, g) \mapsto g \circ f$$

is a continuous map.
Hint: I once promised you that Problem Set 4 #2 would someday be useful. The moment is now.

Now let's focus on maps from a space X to itself. A group G with a topology is called a **topological group** if the maps

$$G \times G \to G : (g, h) \mapsto gh \quad \text{and} \quad G \to G : g \mapsto g^{-1}$$

are both continuous. Common examples include the standard matrix groups $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{GL}(n, \mathbb{C})$ and their subgroups, which have natural topologies as subsets of the vector space of (real or complex) n-by-n matrices. Another natural example to consider is the group

$$\mathrm{Homeo}(X) = \left\{ f \in C(X, X) \;\middle|\; f \text{ is bijective and } f^{-1} \in C(X, X) \right\}$$

for any topological space X, where the group operation is defined via composition of maps. We would like to know what topologies can be assigned to $C(X, X)$ so that $\mathrm{Homeo}(X) \subset C(X, X)$, with the subspace topology, becomes a topological group. Notice that the discrete topology clearly works; this is immediate because all maps between spaces with the discrete topology are automatically continuous, so there is nothing to check. But the discrete topology is not very interesting. Let $\mathcal{T}_H$ denote the topology on $C(X, X)$ with subbase consisting of all sets of the form $\mathcal{U}_{K,V}$ and $\mathcal{U}_{X \setminus V, X \setminus K}$, where again $K \subset X$ can be any compact subset and $V \subset X$ any open subset. Notice that if X is compact and Hausdorff, then for any V open and K compact, $X \setminus V$ is compact and $X \setminus K$ is open, thus $\mathcal{T}_H$ is again simply the compact-open topology. But if X is not compact or Hausdorff, $\mathcal{T}_H$ may be stronger than the compact-open topology.

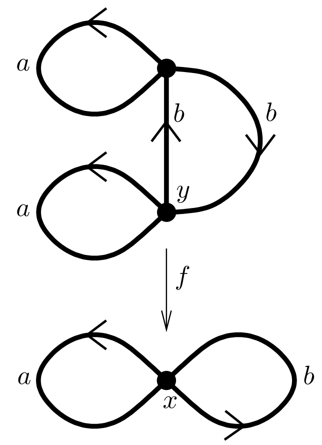
(e) [10 pts] Show that if X is locally compact and Hausdorff, then $\mathrm{Homeo}(X)$ with the topology $\mathcal{T}_H$ is a topological group.
Hint: Notice that $f(K) \subset V$ if and only if $f^{-1}(X \setminus V) \subset X \setminus K$. Use this to show directly that $f \mapsto f^{-1}$ is continuous, and then to reduce the rest to what was proved already in part (d).

[1] Since $C(X \times Y, Z)$ and $C(X, C(Y, Z))$ both have natural topologies in terms of the compact-open topology, you may be wondering whether the correspondence (1) defines a homeomorphism between them. The answer to this is more complicated than one would like, but Steenrod showed in a famous paper in 1967 that the answer is "yes" if one restricts attention to spaces that are *compactly generated*, a property that most respectable spaces have. The caveat is that $C(X, Y)$ in the compact-open topology will not always be compactly generated if X and Y are, so one must replace the compact-open topology by a slightly stronger one that is compactly generated but otherwise has the same properties for most practical purposes. If you want to know what "compactly generated" means and why it is a useful notion, see N. Steenrod, A Convenient Category of Topological Spaces, *Michigan Mathematical Journal* **14** (1967) 133–152. These issues are somewhat important in homotopy theory at more advanced levels, though it is conventional to worry about them as little as possible.

Conclusion: We've shown that if X is compact and Hausdorff, then $\text{Homeo}(X)$ *with the compact-open topology is a topological group. This is actually true under somewhat weaker hypotheses, e.g. it suffices to know that X is Hausdorff, locally compact and locally connected.*[2]

Just for fun, here's an example to show that just being locally compact and Hausdorff is not enough: let $X = \{0\} \cup \{e^n \mid n \in \mathbb{Z}\} \subset \mathbb{R}$ with the subspace topology, and notice that X is neither compact (since it is unbounded) nor locally connected (since every neighborhood of 0 is disconnected). Consider the sequence $f_k \in \text{Homeo}(X)$ defined for $k \in \mathbb{N}$ by $f_k(0) = 0$, $f_k(e^n) = e^{n-1}$ for $n \le -k$ or $n > k$, $f_k(e^n) = e^n$ for $-k < n < k$, and $f_k(e^k) = e^{-k}$. It is not hard to show that in the compact-open topology on $C(X, X)$, $f_k \to \text{Id}$ but $f_k^{-1} \not\to \text{Id}$ as $k \to \infty$, hence the map $\text{Homeo}(X) \to \text{Homeo}(X) : f \mapsto f^{-1}$ is not continuous.

3. [30 pts] The picture below describes a base-point preserving covering map $f : (Y, y) \to (X, x)$, where $X \cong S^1 \vee S^1$, the two generators of $\pi_1(X, x) \cong \mathbb{Z} * \mathbb{Z}$ are labeled by loops a and b based at x, and the preimages of these loops in Y are given the same labels. Let us write elements of $\pi_1(X, x)$ accordingly as words in the letters a and b.



   (a) [10 pts] What is the subgroup $f_* (\pi_1(Y, y)) \subset \pi_1(X, x)$? Describe it as $\langle S \rangle$ or $\langle S \rangle_N$, meaning the smallest subgroup or normal subgroup respectively containing some specific subset $S \subset \pi_1(X, x)$.

   (b) [5 pts] Find a specific element $w \in \pi_1(X, x)$ for which the lifting theorem implies $w \notin f_* (\pi_1(Y, y))$.

   (c) [15 pts] Draw a similar picture to describe a covering map $g : (Z, z) \to (X, x)$ such that $g_* (\pi_1(Z, z)) = \langle a^2, b^2, ab, ba \rangle_N$.
   Hint: First determine what the quotient group $\pi_1(X, x)/\langle a^2, b^2, ab, ba \rangle_N$ is, and use this to deduce the degree of the cover g.[3] *Then consider loops in X based at x, and determine whether their lifts to Z starting at the base point should close up or not.*

[2] If you're interested, R. Arens, Topologies for homeomorphism groups, *Amer. J. Math.* **68** (1946) 593–610 contains a quite clever proof of this fact.

[3] You might want to wait until after the lecture on Friday June 15 before attempting this part.