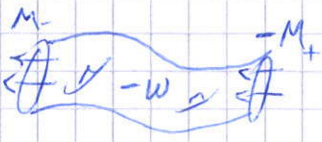
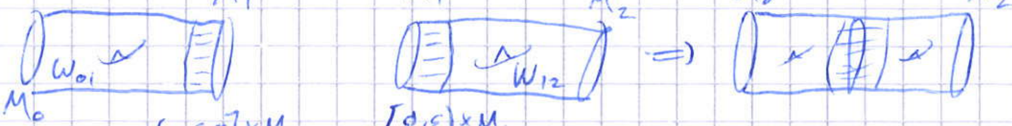
TALK 2: Introduction to bordism & background on fiber bundles (26.4.2018)

oriented bordism: $M_\pm$ closed, smooth oriented $n$-mfds. They are bordant ($M_+ \sim M_-$) if $\exists$ cpt, smooth oriented $(n+1)$-mfd $W$ w/

$\partial W = -M_- \amalg M_+$ 

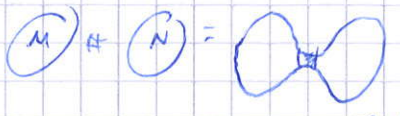
$\sim$ is (i) reflexive:  $[0,1]\times M$ $\Rightarrow M \sim M$, (ii) symmetric: 

(iii) transitive: 

$-M_0$, $W_{01}$, $M_1$, $(-\varepsilon,0]\times M_1$; $-M_1$, $W_{12}$, $M_2$, $[0,\varepsilon)\times M_1$ $\Rightarrow$ $-M_0$, $M_2$

ex: $S^1 \sim S^1 \amalg S^1$  $-S^1$, $S^1\amalg S^1$ or  $-S^1$, $S^1 \amalg S^1$ ($S^1 \sim \emptyset$).

ex: $M \amalg N \sim M \# N$  "1-handle" $[-1,1]\times D^{n-1}$ $-(M\amalg N)$, $M\# N$

(connected sum $M \# N =$  $[M \# N]$)

defn: Abelian group $\Omega_n^{SO} := \left\{ \begin{array}{c} \text{closed oriented} \\ \text{smooth } n\text{-mfds} \end{array} \right\} \Big/ \sim$, $0 = [\emptyset]$, $[M] + [N] := [M \amalg N]$.

ring structure: $\Omega_k^{SO} \otimes \Omega_\ell^{SO} \to \Omega_{k+\ell}^{SO} : [M]\otimes[N] \longmapsto [M\times N]$

$\Rightarrow \Omega^{SO} := \bigoplus_{n\geq 0} \Omega_n^{SO}$ is a ring w/ unit $1 = [pt] \in \Omega_0^{SO}$ $(= \mathbb{Z})$

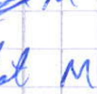
$\exists$ characteristic classes (Pontryagin) $p_k : \left\{ \begin{array}{c} \text{real vector bdls} \\ \text{over } M \end{array} \right\} \to H^{4k}(M)$ for $k \in \mathbb{N}$

s.t. $\forall$ $N \xrightarrow{f} M$, $p_k(f^*E) = f^* p_k(E)$ & $p_k(E \oplus \text{trivial bndl}) = p_k(E)$

$\Rightarrow$ for any partition $n = k_1 + \dots + k_N$, the Pontryagin number of an oriented $n$-mfd

$M \longmapsto \langle \underbrace{p_{k_1}(TM) \cup \dots \cup p_{k_N}(TM)}_{=: \bar{p}(TM)}, [M] \rangle$ defines a hom. $\Omega_n^{SO} \to \mathbb{Z}$ (bordism invt).

Pf that $M = \partial W \Rightarrow \langle \bar{p}(TM), [M] \rangle = 0$.

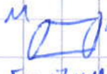
 For inclusion $M \xhookrightarrow{i} W$, $i^* TW = TW|_{\partial W} \cong TM \oplus \underbrace{\text{normal bndl}}_{\text{trivial}}$

$\Rightarrow \bar{p}(TM) = \bar{p}(i^*TW) = i^*\bar{p}(TW) \Rightarrow \langle \bar{p}(TM), [M] \rangle = \langle \bar{p}(TW), \underbrace{i_*[M]}_{=0} \rangle = 0$. ☐

thm (R. Thom 1954): $\exists$ ring iso. $\mathbb{Q}[y_4, y_8, y_{12}, \dots] \xrightarrow{\Phi} \Omega^{SO} \otimes \mathbb{Q}$ w/ generators of deg. $4k$ s.t. $\Phi(y_{4k}) = [\mathbb{CP}^{2k}]$.

thm (C.T.C. Wall 1960): $[M] \in \Omega^{SO}$ is determined by its Pontryagin & Stiefel-Whitney numbers.



defn: Unoriented bordism: $\Omega_n$ = same but ignore orientations, e.g.  $[0,1]\times M$ $\Rightarrow 2[M] = 0$ always in $\Omega_n$.

**fiber bundle**: $F, E, B$ top. spaces, $G$ a top. grp. (or all smooth mfds, $G$ a Lie group...)
"standard fiber" "total space" "base"

$F \hookrightarrow E \xrightarrow[\text{surj}]{\pi} B$ is a fiber bndl if all $x \in B$ have nbhds $x \in \mathcal{U}_\alpha \overset{\text{open}}{\subseteq} B$ admitting local trivializations $E|_{\mathcal{U}_\alpha} := \pi^{-1}(\mathcal{U}_\alpha) \xrightarrow[\text{homeo}]{\Phi_\alpha} \mathcal{U}_\alpha \times F$, $\pi \searrow \mathcal{U}_\alpha \swarrow$ proj.

(if all smooth mfds, homeo $\rightsquigarrow$ diffeo "smooth fiber bndl")

$E_x := \pi^{-1}(x) \cong F$ "fiber over $x$"

$\Gamma(E) := \{ s: B \to E \mid \pi \circ s = \mathrm{Id}, \text{ i.e. } s(x) \in E_x \ \forall x \}$ "sections"

Call $E \xrightarrow{\pi} B$ trivial if $\exists$ a global triv. $E = \pi^{-1}(B) \xrightarrow{\cong} B \times F$.

Local trivs. $\Phi_\alpha, \Phi_\beta \rightsquigarrow$ transition maps $g_{\beta\alpha}: \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{Homeo}(F)$ s.t.

$$\Phi_\beta \circ \Phi_\alpha^{-1} : (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times F \to (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times F$$
$$(x, p) \mapsto (x, g_{\beta\alpha}(x)(p))$$

Say $E \xrightarrow{\pi} B$ has structure group $G$ if $G$ acts on $F$ ($G \xrightarrow{\text{hom.}} \mathrm{Homeo}(F)$) & can fix a collection of local trivs. $\{\Phi_\alpha\}$ covering $B$ s.t. $g_{\beta\alpha}: \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to G$.

Morphism $E \to E'$ (over same base w/ same str. grp): looks like (locally)

$$\mathcal{U}_\alpha \times F \to \mathcal{U}_\alpha \times F' : (x, p) \mapsto (x, f_\alpha(x) p) \text{ for some } f_\alpha: \mathcal{U}_\alpha \to G$$

in local trivs.

ex: $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $F = \mathbb{K}^m$, $G = GL(m, \mathbb{K})$ acting linearly on $\mathbb{K}^m \Rightarrow$
$E \xrightarrow{\pi} B$ is a (real or cpx) vector bundle of rank $m$. Fibers $E_x \cong \mathbb{K}^m$ are vec. spaces, morphisms $E \to E'$ map $E_x \to E'_x$ linearly $\forall x \in B$.
Local trivs. $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{K}^m \Leftrightarrow$ local frames: $\forall x \in \mathcal{U}_\alpha$,

$$s_\alpha(x) := \left(\Phi_\alpha|_{E_x}\right)^{-1} \in \{\text{isos. } \mathbb{K}^m \to E_x\} = \{\text{bases of } E_x\}$$

ex: $M$ a smooth mfd, chart over $\mathcal{U}_\alpha \subseteq M$, $(x^1, \dots, x^n): \mathcal{U}_\alpha \hookrightarrow \mathbb{R}^n \rightsquigarrow$ frame
$\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)$ for $TM|_{\mathcal{U}_\alpha}$: coordinate vector fields, basis of tangent space $T_x M$ $\forall x \in \mathcal{U}_\alpha$.

ex: $F = \mathbb{K}^m$, $G \subseteq GL(m, \mathbb{K})$ a subgroup $\Rightarrow$ V.B. w/ extra structure, e.g.
- $G = O(m)$ or $U(m) \Leftrightarrow$ bndl metric: $\exists$ inner product on $E_x$ $\forall x$, consider only orthonormal frames
- $G = SO(m) \Leftrightarrow$ bndl metric + orientation: oriented orthonormal frames

operations on vector bndls: fix $E, E'$ over same base $B$
- direct sum: $(E \oplus E')_x = E_x \oplus E'_x$, transition maps $\begin{pmatrix} g_{\beta\alpha} & 0 \\ 0 & g'_{\beta\alpha} \end{pmatrix} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to GL(m+m', \mathbb{K}) = \mathrm{Aut}(\mathbb{K}^m \oplus \mathbb{K}^{m'})$
- tensor product: ~~$E \otimes$~~ $E' \to B$ same idea
- complex conjugate: for $\mathbb{K} = \mathbb{C}$, $\bar{E} \to B$ has fibers $\bar{E}_x := E_x$ but scalar mult. ~~by~~

$\mathbb{C} \times \bar{E}_x \to \bar{E}_x : (\lambda, v) \mapsto \bar{\lambda} v$. $\Rightarrow$ transition maps $g_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(m, \mathbb{C})$ for $E$ become $\bar{g}_{\beta\alpha}$ for $\bar{E}$.

prop: $\bar{E} \cong E^*$ dual bundle. pf: Choose bundle metric $\langle , \rangle$, $\bar{E} \xrightarrow{\cong} E^* : v \mapsto \langle v, \cdot \rangle$. ☒

- frame bundle: $E \rightsquigarrow F(E) \to B$, a fiber bundle w/ fibers
$F(E_x) = \{\text{isos. } \mathbb{K}^m \to E_x\} \underset{\text{homeo}}{\cong} GL(m, \mathbb{K})$. Transition maps $g_{\beta\alpha} : U_\alpha \cap U_\beta \to G$ same as for $E$ but now $G$ acts on $F := G$ by left-multiplication.
$\Rightarrow F(E)$ inherits a fiber-preserving right $G$-action $F(E) \times G \to F(E) : (\Phi, g) \mapsto \Phi \circ g$
$G$ acts freely & transitively on each fiber. $=:$ "principal $G$-bundle" (PFB)
(Here $G = GL(m, \mathbb{K})$, but e.g. if $G = O(m)$, $\rightsquigarrow$ principal $O(m)$-bundle w/ fibers
$F(E_x) = \{\text{orthonormal frames of } E_x\}$.)

rk: a V.B. $E \to B$ is trivial $\Leftrightarrow \Gamma(F(E)) \neq \emptyset$. (Note $\Gamma(E) \neq \emptyset$ since e.g. $\exists$ "0-section": $s(x) := 0 \in E_x \ \forall x$)
(For PFBs, sections $\Leftrightarrow$ trivializations:
$s : B \to F(E) \Leftrightarrow \Phi : F(E) \to B \times G$ s.t. $\Phi^{-1}(x, g) = s(x) g$.)

pullback: Given $E \xrightarrow{\pi} B$ & $f : B' \to B$, $\exists$ bundle $f^*E \to B'$ w/ fibers $(f^*E)_x = E_{f(x)}$, transition maps $g_{\beta\alpha} \circ f$.

lemma: If $f_0 \simeq f_1 : B' \to B$ then $f_0^*E \cong f_1^*E$ (bundle iso.)

"smooth pf": $\boxed{B \times I} \underset{f_0}{\overset{f_1}{\xrightarrow{H}}} B$, choose a connection on $H^*E \to B \times I$ s.t. $\forall x \in B$, smooth family of "parallel transport" maps along the path $t \mapsto (x, t) \in B \times I$ ~~$t \mapsto H(x,t) \in B$~~:
$\Phi_x^t : (H^*E)_{(x,0)} \xrightarrow{\cong} (H^*E)_{(x,t)}$, then $\exists$ bundle iso. $f_0^*E \to f_1^*E$
sending $(f_0^*E)_x = E_{f_0(x)} = (H^*E)_{(x,0)} \underset{\cong}{\xrightarrow{\Phi_x^1}} (H^*E)_{(x,1)} = E_{f_1(x)} = (f_1^*E)_x$. ☒

cor: If $B$ is contractible, every fiber bundle over $B$ is trivial. ☒

construction of sections

thm: Suppose $F \hookrightarrow E \xrightarrow{\pi} B$ is a fiber bundle, $B$ a CW-cpx, $A \subseteq B$ a subcpx, $s \in \Gamma(E|_A)$, & $F$ is weakly contractible (i.e. $\pi_k(F) = 0 \ \forall k$). Then: $s$ extends to a section of $E$, which is unique up to htpy of sections fixed on $A$.

pf: Existence: define $s$ arbitrarily on 0-cells not in $A$. Induction: assume $s$ already extended to $(k-1)$-skeleton $B^{k-1} \cup A$. Given a $k$-cell w/ char. map $\Phi_\alpha : \mathbb{D}^k \to B^k$, $\mathbb{D}^k$ contr. $\Rightarrow \Phi_\alpha^*E \cong \mathbb{D}^k \times F$, so section $\Leftrightarrow$ maps $\mathbb{D}^k \to F$, already def'd on $\partial\mathbb{D}^k$ since $\Phi_\alpha(\partial \mathbb{D}^k) \subseteq B^{k-1}$. $\pi_{k-1}(F) = 0 \Rightarrow \exists$ extension to $\mathbb{D}^k \to F$. Uniqueness: Same argument pulled back via $B \times I \to B$. ☒