I

LONG EXACT SEQUENCE FOR FIBER BUNDLES

17-5
2018
PART 2

Reminder 1: Def A fiber bundle is a map $p: E \to B$ ($\hookrightarrow$ projection) such that

- is surjective
- $\forall b \in B$ $\exists$ a local trivialization (i.e. $\exists\, \mathcal{U}_\alpha \subset B$ neighborhood of b s.t.

$$\begin{tikzcd} p^{-1}(\mathcal{U}_\alpha) \arrow[r, "\cong", "\Phi_\alpha"'] \arrow[dr, "p"'] & \mathcal{U}_\alpha \times \underbrace{F}_{:= p^{-1}(b)} \arrow[d] \\ & \mathcal{U}_\alpha \end{tikzcd}$$

$(x,y) \mapsto x$)

Standard notation:

$$F \overset{i}{\hookrightarrow} E \overset{p}{\twoheadrightarrow} B$$

- B $\hookrightarrow$ base
- E $\hookrightarrow$ total space
- F $\hookrightarrow$ standard fiber

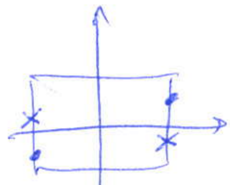
$\leftarrow$ in Hatcher's "Algebraic Topology" he says that is a sort of exact sequence of topological spaces

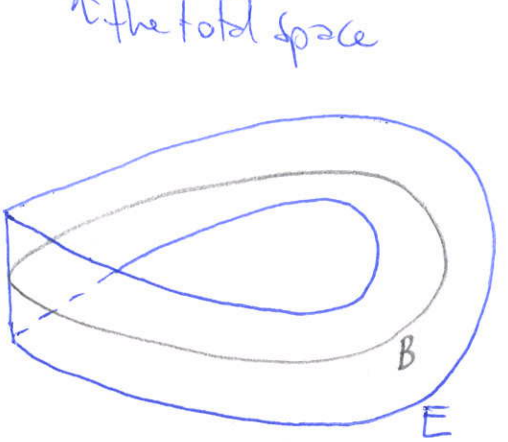
ex 1: Möbius band $\to$ it is a fiber bundle over $B = S^1$

$I := [-1, 1]$

$E = I^2/\sim$ $\quad (1, v) \sim (-1, -v) \quad \forall v \in I$

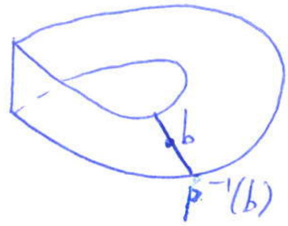
$\uparrow$ the total space





$\forall b \in B \quad p^{-1}(b) = [\{*\} \times I]$ i.e.

$\uparrow$ the fiber



II ex2: klein bottle → similarly, I glue two Möbius bands ⇒ so I get S^1's as fibers.

Def $p: E \to B$ has the <u>homotopy lifting property</u> when,
↖ general
being $g_t: X \to B$ a homotopy, $\tilde{g}_0: X \to E$ lifting of g_0 (i.e. $p\tilde{g}_0 = g_0$), there exists $\tilde{g}_t: X \to E$ lifting g_t

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur] \arrow[r] & B \end{tikzcd}$$

Def When $p: E \to B$ has the homotopy lifting property w.r.t. all spaces X, we call it <u>fibration</u>.

side facts: when E, B path connected, local path connected $p: E \to B$ covering map has the homotopy lifting property with respect to X connected spaces. Furthermore a fiber bundle with a discrete fiber is a covering space (and viceversa)

<u>Theorem 4.41</u>

$p: E \to B$ has the homotopy lifting property w.r.t. disks D^k $\forall k \geq 0$. Then $\forall b_0 \in B$ $\forall x_0 \in F = p^{-1}(b_0) \subset E$ we have that

$p_*: \pi_n(E, F, x_0) \longrightarrow \pi_n(B, b_0)$ is an isomorphism $\forall n \geq 1$

If also B is path connected we have a long exact sequence induced by $F \overset{i}{\hookrightarrow} E \overset{p}{\twoheadrightarrow} B$:

$$\dots \to \pi_n(F, x_0) \xrightarrow{i_*} \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\Phi} \pi_{n-1}(F, x_0) \xrightarrow{i_*} \pi_{n-1}(E, x_0) \to \dots$$

III

Remark: Φ connecting homomorphism has the following definition

$$\Phi : \pi_n(B, b_0) \to \pi_{n-1}(F, x_0)$$

$[f] \in \pi_n(B, b_0) \rightsquigarrow f : (D^n, \partial D^n) \to (B, b_0)$
so to say

for the homotopy lifting property w.r.t. D^k $\forall k \geq 0$ there exists

$\tilde{f} : (D^n, \partial D^n) \to (E, p^{-1}(b_0) = F)$ s.t. $p\tilde{f} = f$

$$\Rightarrow \Phi([f]) = [\tilde{f}|_{\partial D^n}]$$

since $\tilde{f}|_{\partial D^n} : (\partial D^n, *) \to (F, x_0)$

"$\boxed{pf}$" (structure)
- p_* isomorphism: by its definition we prove p_* injective and surjective (see Hatcher, A.T., p 376)
- long exact sequence

STABLE ORTHOGONAL GROUP

recall: $O(n) = \{ A \in \mathbb{R}^{n \times n} : |\det A| = 1 \} = \{ g : \mathbb{R}^n \to \mathbb{R}^n \text{ isometry} \}$

ex $O(1) = \pm 1$

$O(2) = \left\{ \begin{pmatrix} \mp\cos\theta & \sin\theta \\ \pm\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}$

$O(3) = \ldots$

$O(1) \xrightarrow{\sigma} O(2) \xrightarrow{\sigma} O(3) \xrightarrow{\sigma} \ldots$ it is a natural inclusion sequence

$$O := \operatorname{colim}_{q \to \infty} O(q) = \coprod_{q \in \mathbb{N}} O(q) \Big/ \sim$$

Lecture 2 ↑ modulo inclusion maps

IV We would like to see that the induced sequence through π_n functor stabilizes:

Exercise 5.10

$$\pi_n(O(1)) \longrightarrow \pi_n(O(2)) \longrightarrow \pi_n(O(3)) \longrightarrow \dots$$

We take the transitive action of $O(q)$ over $S^{q-1} \subseteq \mathbb{R}^q$

$$O(q) \times S^{q-1} \longrightarrow S^{q-1}$$
$$(A, y) \longmapsto Ay$$

Q: for which $y \in S^{q-1}$ the stabilizer $st(y) = O(q-1) \subset O(q)$?

$\hookrightarrow$ $st(y) \underset{\substack{\uparrow \\ \text{def}}}{=} \{A \in O(q) : Ay = y\} \underset{\substack{\uparrow \\ \text{question}}}{=} O(q-1) \cong$

$\underset{\substack{\uparrow \\ \text{one out of the } q \text{ choices}}}{\cong} \left\{ \left(\begin{array}{c|c} \tilde{A} & \\ \hline & 1 \end{array} \right) \in O(q) : \tilde{A} \in O(q-1) \right\}$

then

$$Ay = y \iff \left(\begin{array}{c|c} \tilde{A} & \\ \hline & 1 \end{array} \right) y = y \iff y = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \text{ the pole}$$

$$\Rightarrow \text{ANSWER: } st\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = O(q-1) \subset O(q)$$

Now fix $x \in S^{q-1}$

$$O(q-1) \overset{i}{\hookrightarrow} O(q) \overset{p}{\twoheadrightarrow} S^{q-1}$$
$$B \longmapsto \left(\begin{array}{c|c} B & \\ \hline & 1 \end{array} \right)$$
$$A \longmapsto Ax$$

using the theorem of the long exact seq. for fiber bundles

chosen $y_0 \in \underbrace{S^{q-1}}_{\text{the base}}$, $B_0 \in \underbrace{O(q-1)}_{\text{the std fiber}} = p^{-1}(y_0)$

V since S^{q-1} is path connected we have:

$$\cdots \to \pi_n(O(q-1), B_0) \xrightarrow{i_*} \pi_n(O(q), B_0) \xrightarrow{p_*} \pi_n(S^{q-1}, y_0) \xrightarrow{\Phi} \pi_{n-1}(O(q-1), B_0) \longrightarrow \cdots$$

Reminder $\pi_n(S^q) = \{0\}$ for $n<q$

$\Rightarrow$ for $n<q-2$ the long exact sequence splits every two terms and we have isomorphisms

then for q sufficiently large $\pi_n(O(q))$ are independent from q, so to say

$$\pi_n(O(1)) \to \pi_n(O(2)) \longrightarrow \pi_n(O(3)) \longrightarrow \cdots$$

stabilizes.

Theorem 5.41 (Bott song)

i mod 8	0	1	2	3	4	5	6	7
$\pi_i(O)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\{0\}$	$\mathbb{Z}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathbb{Z}$

computed in late '50s with Morse theory

ex) FIBER BUNDLES OVER PROJECTIVE SPACES

Reminder $\mathbb{P}^n(\mathbb{K}) := \dfrac{\mathbb{K}^{n+1} \setminus \{0\}}{\mathbb{K} \setminus \{0\}} =$

$= \dfrac{\mathbb{K}^{n+1} \setminus \{0\}}{\sim}$ $\quad v \sim w \overset{\text{def}}{\Leftrightarrow} \exists \lambda \in \mathbb{K}$ s.t. $v = \lambda w$

- $\mathbb{R}$

we have the covering space of index 2

$$p: S^n \longrightarrow \mathbb{P}^n(\mathbb{R}) \cong S^n / x \sim -x$$
$$x \longmapsto [x]$$

$\Rightarrow$ a fiber bundle is $S^0 \longrightarrow S^n \longrightarrow \mathbb{P}^n(\mathbb{R})$

VI

- $\mathbb{C}$

$S^{2n+1} \subset \mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$

$\mathbb{P}^n(\mathbb{C}) \cong S^{2n+1} / z \sim \lambda z \quad \lambda \in S^1$

$$\Rightarrow p : S^{2n+1} \longrightarrow \mathbb{P}^n(\mathbb{C})$$
$$z \longmapsto [z]$$

$\rightarrow$ equivalence class in the quotient

with the S^1's as fibers

(then one proves that $\forall b \in B$ there exists a local trivialization)

$$\Rightarrow S^1 \longrightarrow S^{2n+1} \longrightarrow \mathbb{P}^n(\mathbb{C})$$

this works $\forall n \geq 0$

- $n = \infty \rightarrow$ interesting fiber bundle over $\mathbb{P}^\infty(\mathbb{C})$
- $n = 1$ memo: $\mathbb{P}^1(\mathbb{C}) \cong S^2$

$S^1 \longrightarrow S^3 \longrightarrow S^2$ Hopf bundle (lecture 1)

$S^3 \subseteq \mathbb{C}^2$

$\shortparallel$

$\{(z, w) : |z|^2 + |w|^2 = 1\}$

we consider the action on S^3 of S^1

$$S^1 \times S^3 \longrightarrow S^3$$
$$(\alpha, (z, w)) \longmapsto (\alpha z, \alpha w)$$

$\leftarrow$ still in S^3 since $|\alpha|^2 = 1$

now we want to show that the eq. classes are S^2's

$\hookrightarrow \forall (z, w)$ there is always an $\alpha \in \mathbb{C}$ s.t. $\alpha w \in \mathbb{R}$

(i.e. $\operatorname{Im} w = 0$, so that $(\alpha z, \alpha w) \in S^3 \cap \{\operatorname{Im} w = 0\}$

$\cong$

$S^2 \subset S^3$

VII Another way of seeing it

$$S^3 \longrightarrow \mathbb{C} \cup \{\infty\} \cong S^2$$

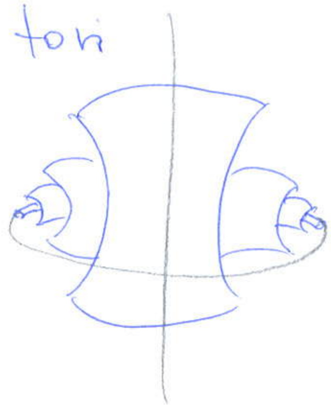
$$(z, w) \longmapsto \frac{z}{w}$$

with $|z|^2 + |w|^2 = 1$ and the fixed ratio $\frac{|z|}{|w|}$

(since we have fibers S^1)

$\Rightarrow$ we are dividing S^3 in many tori

we can see it with $S^3 \cong \mathbb{R}^3 \cup \{\infty\}$



similar reasons

- $\mathbb{H}$

  $S^3 \longrightarrow S^{4n+3} \longrightarrow \mathbb{P}^n(\mathbb{H})$ ($n=1 \rightarrow$ Hopf bundle)

- $\mathbb{O}$

  $n=1$: $S^7 \longrightarrow S^{15} \longrightarrow S^8$