Def (Morphism of G-bundles)

Let $E \xrightarrow{\pi} M$ and $F \to N$ be principal G-bundles. Then a morphism between them is a pair of map $(\tilde{f}, f)$ such that the diagram

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & F \arrow[d, "\pi"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

comutes and $\tilde{f}(p \cdot g) = \tilde{f}(p) \cdot g$.

Lemma If $(\tilde{f}, f)$ is a morphism of free bundles, and f is an homeomorphism, then $(\tilde{f}, f)$ is an isomorphism.

Proof We have just to check fiber-wise the statement. Let $x \in M$. $h, h' \in E_x$ such that $\tilde{f}(h) = \tilde{f}(h')$. Because of transitivity $\exists g \in G$: $h' = h \cdot g$, then.

$\tilde{f}(h') = \tilde{f}(h \cdot g) = \tilde{f}(h) \cdot g = \tilde{f}(h)$ hence $g = e$ and $h' = h$.

Let $\tilde{h} \in F_{f(x)}$. $\tilde{h} = f(h) \cdot g$ for some $h \in E_x$, $g \in G$ and $\tilde{h} = f(h \cdot g)$. $\square$

Pull-back of bundles

Def] Let $F \to N$ be a principal G-bundle $f: M \to N$ continuous. We define

$$f^*F = \bigsqcup_{x\in M} F_{f(x)} \qquad (f^*F)_x := F_{f(x)}$$

$$\begin{tikzcd} f^*F \arrow[d, "\pi"] \\ M \end{tikzcd}$$

with $\pi(h) = x \quad \forall h \in (f^*F)_x$.

An alternative definition is take

$$f^*F = \{(x,h)\in M\times F : f(x) = \pi(h)\}$$

and $\pi(x,h) = x$. We have a natural morphism $\tilde{f}(x,h) = h$

Lemma] (Universal property of pull-back).

Any morphism $(\tilde{f}, f): E \to F$ can be uniquely factorized as $(\tilde{f}, f) = (\tilde{f}', f)\circ(\tilde{f}'', id_M)$

$$\begin{tikzcd} E \arrow[r, "\tilde{f}''"] \arrow[d, "\pi"] \arrow[rr, bend left, "\tilde{f}"] & f^*F \arrow[r, "\tilde{f}'"] \arrow[d, "\pi"] & F \arrow[d] \\ M \arrow[r, "id"'] \arrow[rr, bend right, "f"'] & M \arrow[r, "f"'] & N \end{tikzcd}$$

Just take $\tilde{f}''(h) = (\pi(h), \tilde{f}(h))$. ~~One may~~

~~define universal bundles~~ ~~[illegible]~~ ~~pull-backs using~~

Universal bundle

Def Let G be a Lie group $n \in \mathbb{N}$. $E^u \longrightarrow M^u$ is a (G,n)-universal bundle if the following condition is satisfyied:

$\forall$ ~~P~~ $E \longrightarrow X$ G bundle over X finite CW-complex with $\dim X \leq n$,

$\forall\, Y \subset X$ subcomplex, $\forall$ morphism of bundles

$(\mathcal{F}, f) : E\big|_Y \longrightarrow E^u$, it can be extended to

$(\mathcal{G}, g) : E \longrightarrow E^u$

Remark It's important the case when $Y = \emptyset$. In this case the definition, with the universal property of pull-back, ~~and~~ says that any G-bundle over X can be viewed as pull-back of E^u via an appropriate map.

Example If $G = \{\pm 1\}$, $n = 1$ $E^u \longrightarrow M^u$ is the boundary of the Möbius strip. ~~The~~ Two copies of this is not a (G,n)-universal bundle.

Classification theorem.

$\boxed{\text{Th}}$ Let G be a Lie group. X a CW-complex $\dim X = n$ $E^U \to M^U$ a $(G, n+1)$-universal. $[X, M^U]$ ~~are~~ is the set of maps $X \to M^U$ up to homotopy. $F_G X$ is the set of G-bundles over X up to isomorphism. Then the map.

$$\Phi : [X, M^U] \longrightarrow F_G X$$

given by $\Phi([f]) = [f^* E^U]$ is well defined an bijective.

$\boxed{\text{Proof}}$ Because homotopic maps give isomorphic bundles, Φ is well-defined. It is also surjective for the remark. It remains injectivity.

Let $f_0, f_1 : X \to M^U$ maps s.t. $f_0^* E^U \cong f_1^* E^U$.

Let $(\tilde{g}, \mathrm{id}) : f_0^* E^U \to f_1^* E^U$ be such isomorphism. Let $X' = X \times I$ ~~and~~ and $E' = f_0^* E^U \times I$. Let

$\tilde{F} : f_0^* E^U \times \{0,1\} \longrightarrow E^U$ given by

$\tilde{F}(h, 0) = f_0(h)$ and $\tilde{F}(h, 1) = f_1(\tilde{g}(h))$. Because E^U is $(n+1)$-universal we extend $\tilde{F}$ to $(\tilde{G}, G)$, and G is an homotopy from f_0 to f_1. $\square$

A characterizing property of universal bundles.

Th $E^U \longrightarrow M^U$ is a $(G, n+1)$-universal bundle iff E^U, as total space, satisfies

$$\pi_i(E^U) = 0 \quad \forall\, i = 0 \dots n.$$

Proof $\Rightarrow$. ~~Let~~ Let $\varphi: S^i \longrightarrow E^U$. We consider the trivial bundle $E = S^i \times G$. We have the morphism $(\tilde{f}, f): E \to E^U$ given by $\tilde{f}(x,g) = \varphi(x)\cdot g$ and $f(x) = \pi(\varphi(x))$. We have ~~an extension~~ a bigger bundle $F = D^{i+1} \times G$ such that $E = F|_{S^i}$. The morphism $(\tilde{f}, f)$ can be extended to $(\tilde{g}, g): F \to E^U$. We take $F: D^{i+1} \longrightarrow E^U$ given by $F(x) = \tilde{g}(x, e)$. This proves that $\pi_i(E^U) = 0$.

$\Leftarrow$ Conversely, let $E \to X$ ~~be~~ a bundle $K \subset X$ a sub-complex and $(\tilde{f}, f): E|_K \longrightarrow E^U$ a morphism. We ~~we~~ want extend $\tilde{f}$. To do so we extend $\tilde{f}$ cell by cell, and, by induction, the thesis will follow. Let $\Phi_\alpha: S^i \longrightarrow K$ a gluing map. and $\varphi_\alpha: D^{i+1} \longrightarrow X$ the standard inclusion. We have ~~ar~~ two morphisms $\tilde{\varphi}_\alpha: D^{i+1} \times G \longrightarrow E$ and $\tilde{\Phi}_\alpha: S^i \times G \longrightarrow E$

~~Let $\sigma_\alpha = \mathrm{Im}\, \Phi_\alpha \subset K$. $E|_{\sigma_\alpha}$ has a section given by~~

~~$x \mapsto \tilde{\Phi}_\alpha(\Phi_\alpha^{-1}(x$~~ Let $g: S^i \longrightarrow E^U$ be given by $g(x) = \tilde{f}(\tilde{\Phi}_\alpha(x, e))$. Because $\pi_i(E^U) = 0$ we ~~have~~ can extend ~~f~~ g to $G: D^{i+1} \longrightarrow E^U$. ~~Let~~ $\sigma_\alpha := \mathrm{Im}\, \varphi_\alpha \subset X$. On $E|_{\sigma_\alpha}$ we define $\tilde{\Gamma}(x) = G(\pi(\tilde{\Phi}_\alpha^{-1}(x))) \cdot \pi_{2G}(\tilde{\Phi}_\alpha^{-1}(x))$, and the morphism is extended from K to $K \cup \sigma_\alpha$. $\square$

Construction of universal bundles

We ~~now~~ now construct a universal bundle for the special case $G = GL(k, \mathbb{K})$ where $\mathbb{K}$ can be either $\mathbb{R}$ or $\mathbb{C}$. ~~but We down~~ For simplicity we consider only the case $\mathbb{K} = \mathbb{R}$, as the other one is similar.

Def (~~Stiefel~~ Stiefel manifold) Let $k \in \mathbb{N}$ and H an Hilbert ~~sp~~ space. The space

$$St_k(H) = \{ b \in Hom(\mathbb{R}^k, H) \cong H \oplus \ldots \oplus H : b \text{ is injective} \}$$

with the topology induced by $Hom(\mathbb{R}^k, H)$ is called Stiefel manifold.

Remark $GL(k, \mathbb{R})$ acts on $St_k(H)$ ~~on~~ on the right

$$St_k(H) \times GL(k, \mathbb{R}) \longrightarrow St_k(H)$$
$$(b, g) \longmapsto b \circ g$$

We can now identify $b, b' \in St_k(H)$ when $\exists\, g \in GL(k, \mathbb{R})$ s.t. $b = b' \circ g$. This is clearly an equivalence relation. It is equivalent to saying

$$b \sim b' \quad \text{iff} \quad \operatorname{Im} b = \operatorname{Im} b'$$

We now consider ~~the~~ the quotient space

$$Gr_k(H) := St_k(H)/\sim$$

This is ~~the~~ called grassmannian of k-planes in H.

Set-theoretically $Gr_k(H)$ ~~can~~ be viewed as the set of planes $V \subset H$ such that $\dim V = k$.

Proposition | ~~Th~~ Let $\pi : St_k(H) \to Gr_k(H)$ be the standard fibration. Then

$$St_k(H) \xrightarrow{\pi} Gr_k(H)$$

is a principal $GL(k,\mathbb{R})$-bundle with the standard action. ~~if $\dim H = +\infty$.~~

Theorem | $St_k(H)$ is contractible if $\dim H = +\infty$, and thus

$St_k(H) \to Gr_k(H)$ is a $GL(k,\mathbb{R})$-universal bundle.

To prove the theorem various lemmas are needed.

Lemma 1 | (Tietze Extension Theorem) Let X be a metric space $Y \subset X$ closed $f : Y \to \mathbb{R}$ cont. then $\exists$ $F : X \to \mathbb{R}$ that extends f.

Lemma 2 | Let X be ~~some top~~ a metric space s.t. $\exists Y \subset X$ closed and homeomorphic to $\mathbb{R}$. Then $\exists$ $g : X \to X$ without fixed points.

Proof | Let $l : \mathbb{R} \to Y$ ~~an~~ homeomorphism. Let $f : Y \to \mathbb{R}$ be $f(x) = 1 + l^{-1}(x)$. Extend f to $F : X \to \mathbb{R}$ with Lemma 1. Let $g = l \circ F$. $\square$

~~Th~~ **Lemma 3** | Let H be an Hilbert space s.t. with $\dim H = +\infty$. Let $S_H = \{x \in H : \|x\| = 1\}$. Then S_H is contractible.

Proof | Let $\{v_n\}_{n\in\mathbb{Z}}$ be an orthonormal system.

Let $l : \mathbb{R} \to D$ be piecewise defined as follow.

$$l(t) = \cos\left((t-n)\pi/2\right) v_n + \sin\left((t-n)\frac{\pi}{2}\right) v_{n+1}$$

for $t \in [n, n+1]$. This is an homeomorphism with the image and $\operatorname{Im} l$ is clearly closed. By Lemma 2. we have $g : \cancel{D} \to \cancel{D} \setminus D \to D$ without fixed points.

Using the Hirsch's trick we can prove that S_H and D have the same homotopy type. $\square$

Lemma 4 | $It_1(H)$ is contractible.

Proof | It has the same homotopy type of S_H. $\square$

Proof of the Theorem | Let $\pi : It_k(H) \to It_{k-1}(H)$ be given by the restriction to ~~of~~ $\mathbb{R}^{k-1} \subset \mathbb{R}^k$. That is ~~Each fiber~~ If $b : \mathbb{R}^k \to H$ $\pi(b) : \mathbb{R}^{k-1} \to H$ and $\pi(b)(x) = b(x, 0)$. Each fiber is given by $H \setminus \operatorname{Im} \pi(b)$, ~~wich~~ which is h. equivalent to $\cancel{\pi(b)^\perp}$ $\operatorname{Im} \pi(b)^{\perp} \setminus \{0\}$, and again by Lemma 3. We see that each fiber is contractible.

$\square$

The Universal bundle for faithfully linearly representable Lie group.

We consider now only groups G which can be regarded as subgroup of $GL(k, \mathbb{K})$ with $\mathbb{K}$ either $\mathbb{R}$ or $\mathbb{C}$. By general theory of Lie groups, we know that if G is compact it can be represented in $U(k) \subset GL_{\mathbb{K}}(k, \mathbb{C})$, so this class of groups is relatively wide.

We set $EG = \mathcal{I}t_k(H)$. ~~we can~~ G acts on EG in a natural way provided $G \subset GL(k, \mathbb{R})$. We identify b to b' if $b' = b \cdot g$ with $g \in G$. Let BG the quotient set and $\pi : EG \longrightarrow BG$ the projection.

We have that $EG \xrightarrow{\pi} BG$ is universal G-bundle.

Examples $\{\pm 1\}$, $U(1)$

From ~~G-~~ principal bundles to vector bundles.

~~Let~~ Let $E \to M$ be a $GL(k, \mathbb{R})$-bundle. We can construct an associated vector bundle.

$$V(E) = E \times \mathbb{R}^k / G$$

Where G acts on $E \times \mathbb{R}^k$ in this way.

$$(p, x) = (p \cdot g, g^{-1} x)$$

Conversely if $E \to M$ is a vector bundle. We set $\mathcal{B}(E)_x = \{ b : \mathbb{R}^k \to E_x \text{ injective} \}$.

~~Th~~ The operations V and $\mathcal{B}$ are ~~[illegible]~~ inversion up to isomorphism, and so ~~provi~~ classifying ~~vector~~ $GL(k, \mathbb{R})$ bundles over M classifies automatically the vector bundles over M.

We consider ~~M~~ now the universal subbundle over a grassmanian $Gr_k(H)$.

$$S_k(H) \longrightarrow Gr_k(H)$$

$S_k(H)_V := V$. We note that $\mathcal{B}(S_k(H)) = \mathcal{I}t_k(H)$.

It can be proved that $S_k(H)_V$ is a universal bundle. In the sense that

$$[M, Gr_k(H)] \longleftrightarrow \mathcal{F}_k M$$

is a bijection given by pull-back.