UNIVERSAL CHARACTERISTIC CLASSES

5 min.

recall: G = Lie group, $EG \to BG$ universal principal G-bundle, M = mfld $\Rightarrow \exists$ 1:1 correspondence

$$[M:BG] \xrightarrow{\cong} \{\text{isomorphism classes of principal } G\text{-bundles over } M\}$$

$:=$ homotopy classes of maps $f: M \to BG$

The isomorphism is defined as follows: Let $f: M \to BG$ be a char. map

then $[f] \mapsto [f^* EG]$, where $f^*EG \to M$ is the pullback bundle

recall: X = top. space, C = abelian group $\Rightarrow$ cohom. of X with coeffs in C: $H^*(X;C)$

If R = comm. ring, then the cohomology $H^*(X;R)$ is a $\mathbb{Z}$-graded ring with mult. given by the cup product and it is comm. in a $\mathbb{Z}$-graded sense:

$$\alpha \in H^k(X;R) \quad \beta \in H^e(X;R) \Rightarrow \alpha \smile \beta = (-1)^{k\cdot e} \beta \smile \alpha \in H^{k+e}(X;R)$$

cohomology is homotopy invariant

If $f: X \to X'$ contin. map $\Rightarrow \exists$ induced map on cohom. by pullback

$f^*: H^*(X';C) \to H^*(X;C)$ which depends on f only up to homotopy

Suppose $\alpha \in H^*(BG;C)$, BG = classifying space, $P \to M$ principal G-bundle over M

define $\alpha(P) \in H^*(M;C)$ by: $\alpha(P) := f_P^*(\alpha)$ called a char class of $P \to M$

where $f_P: M \to BG$ any classifying map i.e. $f_P^* EG \cong P$

pullback on cohom.

prop: (EX 7.12) $g: M' \to M$ smooth, $P \to M$ G-bundle. Then $\alpha(g^*P) = g^* \alpha(P)$

$\Longrightarrow$ char. classes are natural

pf: Let $f_P: M \to BG$ be a char. map $\Rightarrow [f_P^* EG] = [P]$ & $\alpha(P) = f_P^*(\alpha) \in H^*(M;C)$

- $\alpha(g^*P) := h^*(\alpha)$, where $h: M' \to BG$ char. map s.t $[h^*EG] = [g^*P]$ (*)
- $g^*(\alpha(P)) = g^*(f_P^*(\alpha)) = (f_P \circ g)^*(\alpha)$

$$[(f_P \circ g)^* EG] = [g^*(f_P^* EG)] \overset{f_P^* EG \cong P}{=} [g^* P] \overset{(*)}{=} [h^* EG]$$

$\Rightarrow$ by thm $[f_P \circ g] = [h] \in [M':BG]$

$\Rightarrow$ they induce the same map on cohomology: $(f_P \circ g)^* = h^*: H^*(BG;C) \to H^*(M';C)$

$\Rightarrow \alpha(g^*P) = h^*(\alpha) = (f_P \circ g)^*(\alpha) = g^*(\alpha(P))$ $\square$

def: cohomology classes in $H^*(BG;C)$ are universal char. classes

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COMPLEX LINE BUNDLES & FIRST CHERN CLASS

$\mathcal{H}$ = cpx. separable Hilbert space
↳ contains a countable, dense subset
↳ a (in this case) complex inner product space that is also a metric space

recall: The classifying space for a cpx. line bundle is the projective space $\mathbb{P}(\mathcal{H})$
From lecture 6 we know, that $\mathbb{P}(\mathcal{H}) \overset{h.e.}{\simeq} \mathbb{CP}^\infty \Rightarrow H^*(\mathbb{P}(\mathcal{H});\mathbb{Z}) \cong H^*(\mathbb{CP}^\infty;\mathbb{Z})$

⇒ from now on write $\mathbb{CP}^\infty$ instead of $\mathbb{P}(\mathcal{H})$

recall from topology $H^*(\mathbb{CP}^\infty;\mathbb{Z}) = \begin{cases} \mathbb{Z} & * \text{ even} \\ 0 & \text{otherwise} \end{cases}$

Let $y \in H^2(\mathbb{CP}^\infty;\mathbb{Z})$ be a generator $\Rightarrow y^k := y \cup \dots \cup y \in H^{2k}(\mathbb{CP}^\infty;\mathbb{Z})$

In particular y^k is a generator of that cohom. group, so we can view the integral cohomology ring as the polynomial ring in y over $\mathbb{Z}$

i.e. $H^*(\mathbb{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z}[y]$ y = gen of $H^2(\mathbb{CP}^\infty;\mathbb{Z})$
$1 \in \mathbb{Z}$ gen. of $H^0(\mathbb{CP}^\infty;\mathbb{Z})$

y is unique up to sign, which we can fix by requiring that

$$\langle y, [\mathbb{CP}^1]\rangle = y(\mathbb{CP}^1) = 1$$

where $[\mathbb{CP}^1] \in H_2(\mathbb{CP}^\infty;\mathbb{Z})$ is the fundamental class of $H_2(\mathbb{CP}^1;\mathbb{Z})$ under the inclusion $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$

recall: $L \to M$ cpx line bundle :⇔ $L \to M$ cpx vector bundle of rank 1, i.e. the fibers L_p ($p \in M$) are cpx vector spaces of dim 1 isomorphic to $\mathbb{C}$

recall: $\exists$ $S \longrightarrow \mathbb{CP}^\infty$ tautological line bundle ($S := \{(\ell, v) \in \mathbb{P}\mathcal{H} \times \mathcal{H} \mid v \in \ell\} \subseteq \mathbb{P}\mathcal{H} \times \mathcal{H}$)

In particular, S is a universal line bundle

def: first Chern class of $S \to \mathbb{CP}^\infty$: $c_1 := -y \in H^2(\mathbb{CP}^\infty)$

This defines the first Chern class for all line bundles over any base since $S \to \mathbb{CP}^\infty$ is a univ. line bundle ⇒ if $L \to M$ is any cpx line bundle $\exists$ char map $f: M \to \mathbb{P}$ s.t $f^*S \cong L \Rightarrow c_1(L) := f^* c_1 \in H^2(M)$

prop: $L_1, L_2 \to M$ cpx line bundles ⇒ $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$

pf: (see Hatcher "Vector bundles & K-theory" prop 3.10, page 86.)

prove this universally

(1) Show $c_1(S_1 \otimes S_2) = c_1(S_1) + c_1(S_2)$, where $S_1 \otimes S_2 \to \mathbb{CP}^\infty \times \mathbb{CP}^\infty$ cpx line bundle with $S_i := p_i^* S$, $p_i : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ projection $S \to \mathbb{CP}^\infty$ univ line bundle

(2) $c_1(S) \in H^2(\mathbb{CP}^\infty)$ generator $\overset{\text{topology}}{\rightsquigarrow}$ $H^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong \mathbb{Z}[\alpha_1, \alpha_2]$

$\alpha_i := p_i^*(c_1(S)) = c_1(p_i^* S) = c_1(S_i)$

(3) $i : \mathbb{CP}^\infty \vee \mathbb{CP}^\infty \hookrightarrow \mathbb{CP}^\infty \times \mathbb{CP}^\infty$ induces an iso on H^2 look at 2-skeleton of cw-cpx. $\mathbb{CP}^1_1 \to \mathbb{CP}^1 \times \{pt\}$
$\mathbb{CP}^1_2 \to \{pt\} \times \mathbb{CP}^1$

⇒ to compute $c_1(S_1 \otimes S_2)$ it suffices to compute $i^* c_1(S_1 \otimes S_2)$
$= c_1(i^*(S_1 \otimes S_2))$

(4) Over the first copy of $\mathbb{CP}^\infty$ in the wedge product S_2 is the trivial line bundle so the restriction of $S_1 \otimes S_2$ to this $\mathbb{CP}^\infty$ is $S_1 \otimes 1 \cong S_1$

Similarly $S_1 \otimes S_2$ restricts to S_2 over the second $\mathbb{CP}^\infty$

$\Rightarrow c_1(S_1 \otimes S_2)$ restricts to $\mathbb{CP}^\infty \vee \mathbb{CP}^\infty$ as $\alpha_1 + \alpha_2$ restricted to $\mathbb{CP}^\infty \vee \mathbb{CP}^\infty$

$\Rightarrow c_1(S_1 \otimes S_2) = c_1(S_1) + c_1(S_2)$

(5) the general case follows by considering char maps $f_i : M \to \mathbb{CP}^\infty$ s.t. $L_i \cong f_i^* S$ $\Rightarrow$ for $F = (f_1, f_2) : M \to \mathbb{CP}^\infty \times \mathbb{CP}^\infty$ we have

$F^*(S_i) \cong f_i^* S \cong L_i$ and we use the naturality of the char. classes to get

$$c_1(L_1 \otimes L_2) = c_1(F^* S_1 \otimes F^* S_2) = c_1(F^*(S_1 \otimes S_2)) = F^* c_1(S_1 \otimes S_2)$$
$$= F^*(c_1(S_1) + c_1(S_2)) = F^* c_1(S_1) + F^* c_1(S_2) = c_1(F^* S_1) + c_1(F^* S_2)$$
$$= c_1(L_1) + c_1(L_2)$$ □

cor: $L \to M$ cpx line bundle $\Rightarrow$ $c_1(\underbrace{L^*}_{\text{dual bundle}}) = -c_1(L)$

pf: $P := \{pt\}$ = one point space, $E \to P$ trivial cpx line bundle

$f : M \to P$ map sending all points of M to pt $\Rightarrow$ $f^* E \to M$ trivial line bundle

$\{(m,e) \in M \times E \mid f(m) = \pi(e)\} = M \times E$

$L^* \otimes L \cong$ triv. line bundle $= f^* E$

$$\begin{aligned} \Rightarrow c_1(L^* \otimes L) &\overset{\downarrow}{=} c_1(f^* E) = f^* \underbrace{c_1(E)}_{\in H^2(P) = \{0\}} = 0 \\ &\| \\ c_1(L^*) &+ c_1(L) \end{aligned}$$ □

THE LERAY-HIRSCH THEOREM

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thm: (Leray-Hirsch): $F \hookrightarrow E \to B$ fiber bundle, R = comm. ring

Suppose $\exists\, \alpha_1, \dots, \alpha_N \in H^*(E, R)$ s.t. $i^*\alpha_1, \dots, i^*\alpha_N$ freely generate the R-module $H^*(F; R)$ $\quad i : F \hookrightarrow E$ incl. (In part. $H^n(F;R)$ = fin. gen. R module $\forall n$)

Then $H^*(E,R)$ is isomorphic to the free $H^*(B;R)$-module w/ basis $\alpha_1, \dots, \alpha_N$

$b \cdot c := \pi^*(b) \cup c$ for $b \in H^*(B;R)$ $c \in H^*(E;R)$

An equivalent formulation of the statement is: $\Phi : H^*(B;R) \otimes H^*(F;R) \xrightarrow{\cong} H^*(E;R)$

$$\sum_{ij} b_i \otimes i^*\alpha_j \longmapsto \sum_{ij} \pi^*(b_i) \cup \alpha_j$$

is an R-module iso.

! Φ need not be a ring iso!

remark: In other words the theorem states that under those conditions even thoug E is not isomorphic to $B \times F$, the cohomology of E behaves as though they are.

HATCHER AL.TOP 3.15: $H^n(F;R)$ is a $\overset{\text{fin. gen}}{\text{free}}$ R-module $\forall n \Rightarrow H^*(B;R) \otimes H^*(F;R) \xrightarrow{\cong} H^*(B \times F; R)$

pf: (see Hatcher's Algebraic topology, Theorem 4D.1, page 432) □

example of a fiber bundle for which the hypothesis are not satisfied

Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$: $H^*(S^n) = \begin{cases} \mathbb{Z} & * = 0, n \\ 0 & \text{otherwise} \end{cases}$

$\nexists\ \alpha \in H^1(S^3) = \{0\}$ s.t $\underbrace{i^*\alpha}_{=0} \in H^1(S^1)$ generator

$\Rightarrow \nexists\ \alpha_1, \dots, \alpha_N \in H^*(S^3)$ s.t $i^*\alpha_1, \dots, i^*\alpha_N$ freely generate $H^*(S^1)$

And as a result:

$$\bigoplus_{i=0}^{1} H^i(S^2) \otimes H^{1-i}(S^1) = \mathbb{Z}\otimes\mathbb{Z} \oplus 0 \otimes \mathbb{Z} \cong \mathbb{Z} \neq \{0\} = H^1(S^3)$$

$$\Rightarrow \quad H^*(S^2) \otimes H^*(S^1) \ncong H^*(S^3)$$

FLAG BUNDLES & HIGHER CHERN CLASSES 17 15:40

E = cpx. vector space, $\dim E = k$, has a hermitian metric

$\mathbb{F}(E)$ = associated flag mfld. The points of $\mathbb{F}(E)$ are orthogonal decompositions

$E = \ell_1 \oplus \dots \oplus \ell_k$ of E as a sum of lines

$\mathbb{F}(E) := \{ \underbrace{\ell_1 \oplus \dots \oplus \ell_k}_{=E} \mid \ell_1 \perp \dots \perp \ell_k \}$

$\exists\ k$ tautological line bundles $L_j \to \mathbb{F}(E)$ $j = 1, \dots, k$

The fiber of L_j at $\ell_1 \oplus \dots \oplus \ell_k \in \mathbb{F}(E)$ is ℓ_j

$\rightsquigarrow$ for families: $E \to M$ hermitian vector bundle of rank k over a smooth mfld M $\Rightarrow \exists$ associated fiber bundle - the flag bundle

$$\pi : \mathbb{F}(E) \to M$$

with fibers $(\mathbb{F}(E))_p := \mathbb{F}(E_p)$ the flag mfld of the fiber E_p

$\exists\ k$ taut. line bundles $L_j \to \mathbb{F}(E)$ $j = 1, \dots, k$

prop: The cohomology $H^*(\mathbb{F}(E); \mathbb{Z})$ is a free $H^*(M; \mathbb{Z})$-module with basis all monomials of the form $y_1^{i_1} \cdots y_{k-1}^{i_{k-1}}$ $y_j := c_1(L_j^*) = -c_1(L_j)$ and $0 \le i_j \le k - j$

sketch of proof: (for complete pf see Bott - Differential forms in algebraic topology §21)

- Consider the projective bundle $\pi_1 : \mathbb{P}(E) \to M$, $(\mathbb{P}(E))_p = \underset{\cong\,\mathbb{C}P^{k-1}}{\mathbb{P}(E_p)}$ $p \in M$

  $\exists$ taut. line bundle $S \to \mathbb{P}(E)$ which restricts on each fiber $\mathbb{P}(E)_p$ as the taut. line bundle on that proj. space $L \to \mathbb{C}P^{k-1}$

- $H^*(\mathbb{P}(E)_p; \mathbb{Z}) = H^*(\mathbb{P}(E_p); \mathbb{Z}) = H^*(\mathbb{C}P^{k-1}; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^n)$

  where $\alpha \in H^2(\mathbb{C}P^{k-1}; \mathbb{Z})$ generator $\alpha = c_1(L^*) = -c_1(L)$

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- take $y = c_1(S^*) \in H^2(\mathbb{P}(E);\mathbb{Z})$, $i: \mathbb{C}P^{k-1} \hookrightarrow \mathbb{P}(E)$ incl. of the fiber

$\Rightarrow i^* y = i^* c_1(S^*) = c_1(i^* S^*) = c_1(L^*) = \alpha$

$\Rightarrow$ the pullback of $1, y, \dots, y^{k-1} \in H^*(\mathbb{P}(E);\mathbb{Z})$ via i^* to $H^*(\mathbb{C}P^{k-1};\mathbb{Z})$

freely generate the cohom. of the fiber

$\xrightarrow{\text{Leray - Hirsch}}$ $H^*(\mathbb{P}(E);\mathbb{Z}) \cong H^*(M;\mathbb{Z})\{1, y, \dots, y^{k-1}\}$ *

- Now take the quot. bundle Q given as the orth. compl of S in the triv. bundle over $\mathbb{P}(E)$ $\quad (0 \to S \to \mathbb{P}(E) \times \mathbb{C}^k \to Q \to 0$ ex.)

Consider the proj bundle assoc. to $Q \to \mathbb{P}(E)$: $\mathbb{P}(Q) \to \mathbb{P}(E)$

repeat

$$\begin{array}{ccccccccc}
\overset{L_1}{\overset{\shortparallel}{S_1}} \oplus \dots \oplus \overset{L_{k-1}}{\overset{\shortparallel}{S_{k-1}}} \oplus \overset{L_k}{\overset{\shortparallel}{Q_{k-1}}} & & \cdots & & S_1 \oplus S_2 \oplus Q_2 & & S_1 \oplus Q_1 & & E \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{F}(E) = \mathbb{P}(Q_{k-2}) & \to & \cdots \to \mathbb{P}(Q_2) & \to & \mathbb{P}(Q_1) & \to & \mathbb{P}(E) & \to & M
\end{array}$$

$\Rightarrow H^*(\mathbb{F}(E);\mathbb{Z}) \cong H^*(M;\mathbb{Z})\{ y_1^{i_1} \cdots y_{k-1}^{i_{k-1}} \mid y_j = c_1(L_j^*) = c_1(S_j^*) \quad 0 \le i_j \le n-j \}$

higher Chern classes take $y^k \in H^{2k}(\mathbb{P}(E);\mathbb{Z})$

because of * $\exists!\, c_i(E) \in H^{2i}(M;\mathbb{Z})$ $\quad i = 1, \dots, k$ s.t.

$$y^k + \underbrace{c_1(E) \cdot y^{k-1}}_{= \pi_1^*(c_1(E)) \cup y^{k-1}} + \dots + c_{k-1}(E) \cdot y + c_k(E) = 0$$

$c_i(E)$ is called the i^{th} Chern class of $E \to M$

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$\exists!$ seq. of functions $c_1, c_2, \dots$ assigning to each cpx. vector bundle $E \to M$ a class $c_i(E) \in H^{2i}(M;\mathbb{Z})$ depending only on the isomorphism type of E s.t.

(1) $c_i(f^* E) = f^* c_i(E)$ for the pullback f^* $\quad (f: M' \to M)$

(2) $c(E_1 \oplus E_2) = c(E_1) \cup c(E_2)$ $\quad c = 1 + c_1 + c_2 + \dots \in H^*(M;\mathbb{Z})$ total chern class

(3) $c_i(E) = 0$ $\quad$ if $i > k =$ rank of E

(4) For the taut. line bundle $S \to \mathbb{C}P^\infty$, $c_1(S)$ is a gen. of $H^2(\mathbb{C}P^\infty;\mathbb{Z})$ specified in advance