Recall: $E \to M$ complex vector bundle

There exists $\pi: F(E) \to M$ a flag bundle of E. Prop 7.29 says that $H^*(F(E))$ is a free $H^*(M)$-module. The immediate corollary of 7.29 is:

$\boxed{\text{Cor. 7.30}}$: $\pi^*: H^*(M) \to F(E)$ is injective.

$\boxed{\text{Prop 7.38}}$: The image of Chern class $c_i(E)$ under π^* is the ith elementary symmetric polynomial in $c_1(L_1), c_1(L_2), \dots, c_1(L_k)$, where $L_j \to F(E)$ are the jth tautological line bundle of $F(E)$.

Proof: Consider maps $p_j : F(E) \to P(E)$

$$l_1 \perp l_2 \perp \dots \perp l_k \to l_j$$

$\rightsquigarrow$ line bndl

$$\begin{tikzcd} p_j^*S \cong L_j \arrow[r] \arrow[d] & S \arrow[d] \\ F(E) \arrow[r, "p_j"'] & P(E) \end{tikzcd}$$

, where S is a universal (line) subbundle

$$\begin{tikzcd} S \subset f^*E \arrow[r] \arrow[d] & E \arrow[d] \\ P(E) \arrow[r] & M \end{tikzcd}$$

By naturality of Chern classes, $p_j^* c_i(S^*) = c_i(p_j^* S^*) = -c_i(L_j) =: -x_j$.
Since $c_1(S^*)$ satisfies the defining polynomial $Y^k + c_1(E)Y^{k-1} + \dots + c_k(E) = 0$, so does $-x_j$ $\Rightarrow$ we have precisely k roots of poly degree k

$\Rightarrow Y^k + \pi^* c_1(E) Y^{k-1} + \dots + \pi^* c_k(E) = \prod (Y + x_j)$

$\Rightarrow \pi^* c_i(E) = \sum\limits_{1 \le j_1 < \dots < j_i \le k} x_{j_1} \cdots x_{j_i}$ an elementary symmetric poly. ☒

Note: Since any symmetric poly in $x_j = c_1(L_j)$, $j = 1 \dots k$, is a poly in elementary symmetric poly's, it is automatically a poly in Chern classes.

Prop 7.38 and injectivity of $\pi^*: H^*(F(E)) \to H^*(M)$ allows us to formulate the splitting principle:
To prove a poly identity in the Chern classes of complex v. bndl, it suffices to prove it under the assumption that the v. bndls are direct sum of line bndls.

Def Total chern class $C(E) := 1 + c_1(E) + c_2(E) + \dots$

Total Chern class is, again, natural for pullbacks since $f^*(c_k(E)) = 0$ is

$$f^*(\underset{\substack{\| \\ c_1(S^*)}}{y}^k + c_1(E)y^{k-1} + \dots + c_k(E)) = \underset{\substack{\| \\ c_1(f^*S^*)}}{\tilde{y}}^k + f^*c_1(E)\tilde{y}^{k-1} + \dots + f^*c_k(E) = 0 \text{ is}$$

again a defining poly for Chern classes of f^*E.

Ex 1 $\pi^* C(E) = \prod_{j=1}^{k}(1 + c_1(L_j)) = \prod_{j=1}^{k} c(L_j)$ since $\pi^*(c_1(E) + c_2(E) + \dots + c_k(E)) + 1 = (\sum_{i=1}^{k} i\text{th symm poly}) + 1 = \prod_{i=1}^{k}(1 + x_i)$

Ex 2 **Whitney sum formula**: If $E_1, E_2 \to M$ v. bndls, then $\boxed{C(E_1 \oplus E_2) = C(E_1)C(E_2)}$

This is an important example that demonstrates the use of the splitting principle.

Proof First suppose $E_1 \simeq L_1^1 \oplus \dots \oplus L_k^1$ and $E_2 \simeq L_1^2 \oplus \dots \oplus L_\ell^2$.

Then, by prop 7.38, $C(E) = \prod_{i=1}^{k} C(L_i^1) \prod_{i=1}^{\ell} C(L_i^2) = C(E_1)C(E_2)$.

In general case, by naturality, $\pi^* C(E_1 \oplus E_2) = c(\pi^*(E_1 \oplus E_2)) =$

$= c(\oplus L_i^1) c(\oplus L_i^2) = C(\pi^* E_1) C(\pi^* E_2) =$

$= \pi^* C(E_1) C(E_2)$.

So $C(E_1 \oplus E_2) = C(E_1)C(E_2)$ since π^* is injective. $\square$

What we have now defined satisfies a set of axioms.

Axioms of Chern classes (as in Hatcher VB Thm 3.2) $E \to M$ complex v. bndl.

1. (naturality) $c_i(f^*(E)) = f^*(c_i(E))$ for a pullback $f^*(E)$
2. (Whitney sum) $c(E_1 \oplus E_2) = c(E_1) \cup c(E_2)$
3. $c_i(E) = 0$ if $i > \dim E$.
4. (normalization) for universal line bndl $S \to \mathbb{C}P^\infty$, $c_1(S)$ is a generator of $H^2(\mathbb{C}P^\infty; \mathbb{Z})$ specified in advance.

Normalization condition ~~precludes~~ defining $c_i(E) = 0$ for all E; it also precludes setting $c_1(S) = k\alpha$, where $\alpha \in \mathbb{Z}[\alpha] \simeq H^*(\mathbb{C}P^\infty; \mathbb{Z})$, for $|k| > 1$.

Chern classes defined via these axioms are unique: normalization axiom determines chern classes for universal line bndl (along with property (3), we get $C(S) = 1 + c_1(S)$); naturality determines $C(L)$ for all line bndls L; Whitney sum gives chern classes for all direct sums of line bndls. To show that we now have Chern classes for all bndls we use the splitting principle.

$\pi: F(E) \to M$

Ex 3. Tensor product:

Recall, $c_1(L \otimes L') = c_1(L) + c_1(L')$.

Then $\pi^* c(E \otimes F) = c(\oplus L_i^1 \otimes \oplus L_j^2) = c(\oplus L_i^1 \otimes L_j^2) = \prod c(L_i^1 \otimes L_j^2) =$

$= \prod (1 + c_1(L_i^1) + c_1(L_j^2))$

In particular, $c(E \otimes L) = \prod (1 + c_1(L_i) + c_1(L)) = \sum_{i=0}^{k} c_i(E)(1 + c_1(L))^{n-i}$

L: line bndl.

Ex 4 Stability.

Let $E \to M$ m-dim v. bndl. If we do a direct sum w/ trivial bndl we get $c(E \oplus (\mathbb{C}^n \times M)) = c(E)\, c(\underbrace{\mathbb{C}^n \times M}_{=1}) = c(E)$.

We say chern classes are stable.

This is bad news since we are not able to distinguish stably trivial vector bndls.

Ex 5 Conjugate bndl. Given v. bndl $E \to M$ with Hermitian metric, we get a conjugate bndl by redefining scalar multiplication as $\underbrace{\lambda(v)}_{\text{in } \bar{E}} := \underbrace{\bar{\lambda} v}_{\text{as in } E}$, where $\lambda \in \mathbb{C}$, $v \in E$. The structure group of E is a unitary group $U(n)$, hence the structure group of $\bar{E}$ is again $U(n)$, but we take conjates of the transition matrices. Since for $\bar{g}_{\alpha\beta} \in U(n)$, $\bar{g}_{\alpha\beta} = (g_{\alpha\beta}^T)^{-1}$ is also a transition matrix for E^* (dual bndl), we conclude that $\bar{E} \simeq E^*$. Now we use that to compute $\bar{E}$.

$\pi^* c(E) = \prod (1 + c_1(L_i))$, $\quad \pi^* c(E^*) = \prod (1 + c_1(L_i^*)) = \prod (1 - c_1(L_i))$

$\Rightarrow c_i(E^*) = (-1)^i c_i(E)$.

Ex 6 Computation of tangent bndl $T\mathbb{C}P^n$ of $\mathbb{C}P^n$.

We have split exact sequence $0 \to S \to \mathbb{C}P^n \times \mathbb{C}^{n+1} \to Q \to 0$

S: line bndl; Q: quotient bndl.

Identify $T\mathbb{C}P^n \simeq \mathrm{Hom}(S, Q) \simeq Q \otimes \underbrace{\mathrm{Hom}(S, \mathbb{C})}_{= S^*}$; point $v_\ell \in T\mathbb{C}P^n$ can be identified with a motion of ℓ; this corresponds to a linear map from ℓ to Q.

Tensoring now the sequence $0 \to \underbrace{S \otimes S^*}_{\text{trivial}} \to \mathbb{C}P^n \times \mathbb{C}^{n+1} \otimes S^* \to \underbrace{Q \otimes S^*}_{\simeq T\mathbb{C}P^n} \to 0$

$\Rightarrow c(T\mathbb{C}P^n) = c(Q \otimes S^*) = c(Q \otimes S^* \oplus S \otimes S^*) = c(\mathbb{C}P^n \times \mathbb{C}^{n+1} \otimes S^*) = c(\overset{n+1}{\oplus} S^*) = (1 + c(S^*))^{n+1}$

tensor product properties

Alternatively, since $Q \oplus S$ is trivial, we have

$$c(Q) = \frac{1}{c(S)} = \frac{1}{1+c_1(S)} = \frac{1}{1-\underbrace{c(S^*)}_{= x}} = 1+x+x^2+\ldots+x^n .$$

$$\Rightarrow c(T\mathbb{C}P^n) = c(Q \otimes S^*) = \sum_{i=1}^{n} c_i(Q)(1+\underbrace{c_1(S^*)}_{= x})^{n-i} = \sum_{i=1}^{n} x^i (1+x)^{n-i} =$$

$$= \ldots = (1+x)^{n+1}$$