Euler Class and Intersection Theory

Seminar "Bordism Theory"

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1 Introduction

In this talk we show further properties of the Euler Class $e(E) \in \mathrm{H}^n(M)$ of an oriented rank n vector bundle $E \to B$. Recall that it is defined as the image of the Thom class $u \in \mathrm{H}^n(E, E_0)$ under the composition $\mathrm{H}^n(E, E_0) \to \mathrm{H}^n(E) \to \mathrm{H}^n(B)$ i.e.

$$e(E) := \pi^{-1^*}(u|_E) \in \mathrm{H}^n(B)$$

where $\cdot|_E \colon \mathrm{H}^n(E, E_0) \to \mathrm{H}^n(E)$ is the restriction map.

Theorem 1.1. *Let $E \to M$ a smooth oriented rank n real vector bundle over a closed oriented manifold M. Let ψ be a section which intersects the zero section transversely and let $Z = \psi(M) \cap M$ where M is identified with the zero section of E. Then its Euler class is the Poincare Dual of the fundamental class of Z:*

$$e(E) = [Z]^* \in \mathrm{H}^n(M; \mathbb{Z})$$

Theorem 1.2. *Let M be a compact oriented n-manifold. Then its Euler characteristic is*

$$\chi(M) = \langle e(M), [M] \rangle$$

To give proofs for these results we will make a quick digression into intersection theory.

2 Intersection Theory

Let X a closed oriented smooth manifold of dimension n and A and B oriented smooth submanifolds of X of dimension $n - i$ and $n - j$ respectively. Assume that A and B intersect transversely so that for every $p \in X$ the map $T_pA \oplus T_pB \to T_pX$ induced by the inclusions is surjective. Then $A \cap B$ is a submanifold of dimension $n - (i + j)$.

We give an orientation of $T_pA \cap B$ by the following convention: Choose an oriented basis $u_1, \dots, u_{n-i-j}, v_1, \dots, v_j, w_1, \dots, w_i$ of T_pX such that $u_1, \dots, u_{n-i-j}, v_1, \dots, v_j$ is an oriented basis of T_pA and $u_1, \dots, u_{n-i-j}, w_1, \dots, w_i$ is an oriented basis of T_pB. Then the orientation of $T_pA \cap B$ is given by $u_1, \dots, u_{n-i-j}$. In particular, if $i + j = n$ so that $A \cap B$ is a finite set of points, then p is positively oriented if the map $T_pA \oplus T_pB \to T_pX$ preserves orientation. In these notes we denote this sign by $I(A, B, p)$.

Recall that if M is compact and oriented then the Poincaré duality isomorphism $\mathrm{H}^i(M, \mathbb{Z}) \cong \mathrm{H}_{n-i}(M, \mathbb{Z})$ is given by $\alpha \mapsto [M] \frown \alpha$ with inverse $\beta^* \leftarrow\!\shortmid \beta$.

Theorem 2.1. *The cup product is Poincaré dual to intersection:*

$$[A]^* \smile [B]^* = [A \cap B]^* \in \mathrm{H}^{i+j}(X; \mathbb{Z})$$

Remark. Note that not all homology classes of a closed oriented manifold can be represented as the fundamental class of an oriented submanifold.

Definition 2.2. For a closed oriented manifold X of dimension n we define the intersection pairing

$$\begin{aligned} \cdot \colon\ & \mathrm{H}_{n-i}(X) \otimes \mathrm{H}_{n-j}(X) \to \mathrm{H}_{n-i-j}(X) \\ & \alpha \cdot \beta := [X] \frown (\alpha^* \smile \beta^*) = \alpha \frown \beta^*. \end{aligned}$$

For cohomology classes of complementary dimension, i.e. $i + j = n$, we will often understand $\alpha \cdot \beta \in \mathrm{H}_0(X, \mathbb{Z})$ as an element of $\mathbb{Z}$ by replacing it with $\langle 1, \alpha \cdot \beta \rangle$.

We require a lemma to prove the theorem. Let $E \to B$ be a vector bundle with a bundle metric, let D be its unit disk bundle and ∂D its unit sphere bundle. Note $\mathrm{H}_*(D_0, \partial D) = 0$ so we have an isomorphism $\mathrm{H}_*(D, \partial D) \cong \mathrm{H}_*(D, D_0)$ and we can interpret the Thom class as an element $u \in \mathrm{H}^n(D, \partial D) \cong \mathrm{H}^n(D, D_0) \cong \mathrm{H}(E, E_0)$ where the latter isomorphism is given by excision.

Lemma 2.3. *Let B be a closed oriented smooth k-manifold and let E be a rank n oriented real vector bundle over B with unit disk bundle D. Then*

$$(i_B^D)_*[B] = [D] \frown u \in \mathrm{H}_k(D)$$

where $[B] \in \mathrm{H}_k(M)$ and $[D] \in \mathrm{H}_{n+k}(D, \partial D)$ are the fundamental classes of the manifolds and i_B^D is the zero section of D.

Here we give E any metric. The orientation of D is determined by the orientation of the fiber and the base, in that order.

Proof (Sketch). Let B connected, then we have isomorphisms

$$\mathbb{Z} = \mathrm{H}^0(B; \mathbb{Z}) \xrightarrow{\pi^*(\cdot) \smile u} \mathrm{H}^n(D, \partial D; \mathbb{Z}) \xrightarrow{[D] \frown \cdot} \mathrm{H}_k(D) \xrightarrow{\pi_*} \mathrm{H}_k(B) = \mathbb{Z}$$

The generator $1 \in \mathrm{H}^0(B,\mathbb{Z})$ maps to $[D] \frown u \in \mathrm{H}_k(D)$ and to a generator of $\mathrm{H}_k(B)$ so we obtain $[D] \frown u = \pm(i_B^D)_*[B]$, however there might be a sign depending on n or k. We do not show explicitly how the sign can be nailed down. Using the defining property of the fundamental class $[B]$ one can reduce to the case where B is trivial. Then the equation can be verified explicitly singular chains and cochains associated to oriented bases. Alternatively one can use real coefficients and use de Rham cohomology, so the sign is easily understood in terms of the orientations. $\square$

Now in the setting of the main theorem let $N_A^X = N$ be a tubular neighborhood of A, so that the normal bundle of A is isomorphic to N_A^X by a diffeomorphism which maps the zero section to A. We can interpret the Thom class as an element $u_A \in \mathrm{H}^i(N, N \setminus A; \mathbb{Z})$ and by the lemma, $(i_A^N)_*[A] = [N] \frown u_A$. Consider the maps

$$\mathrm{H}^i(N, N \setminus A; \mathbb{Z}) \to \mathrm{H}^i(X, X \setminus A; \mathbb{Z}) \to \mathrm{H}^i(X; \mathbb{Z})$$

where the first map is given by excision and the second is given by restriction. Denote the image of u_A by $u_A^X \in \mathrm{H}^i(X;\mathbb{Z})$.

Lemma 2.4.

$$\left((i_A^X)_*[A]\right)^* = u_A^X$$

or equivalently

$$(i_A^X)_*[A] = [X] \frown u_A^X \in \mathrm{H}_{n-1}(X)$$

Now we can prove the theorem

Proof of 2.1. We are given an orientation of X, A, B and $A \cap B$. Thus we can define orientations of the normal bundles N_A^X, N_B^X, $N_{A\cap B}^X$ and $N_{A\cap B}^A$ according to the “fiber first” convention so that we can use Lemma 2.4 without any sign. Note there is a canonical isomorphism of vector bundles

$$N_{A\cap B}^A = (N_B^X)|_{A\cap B}$$

which preserves orientations. Thus

$$u_{A\cap B}^A = (i_A^X)^* u_B^X$$

By the lemma it suffices to show that $u_A^X \smile u_B^X = u_{A\cap B}^X$ or equivalently

$$[X] \frown u_{A\cap B}^X = [X] \frown \left(u_A^X \smile u_B^X\right)$$

We compute

$$\begin{aligned} [X] \frown u_{A\cap B}^X &= \left(i_{A\cap B}^X\right)_* [A \cap B] \\ &= \left(i_A^X\right)_* \left(i_{A\cap B}^A\right)_* [A \cap B] \end{aligned}$$

$$
\begin{aligned}
&= \left(i_A^X\right)_* \left([A] \frown u_{A\cap B}^A\right) \\
&= \left(i_A^X\right)_* \left([A] \frown (i_A^X)^* u_B^X\right) \\
&= \left(\left(i_A^X\right)_* [A]\right) \frown u_B^X \\
&= \left([X] \frown u_A^X\right) \frown u_B^X \\
&= [X] \frown \left(u_A^X \smile u_B^X\right)
\end{aligned}
$$

□

3 The Euler Class

Recall that the Euler Class $e(E)$ of an oriented rank n vector bundle E is defined as

$$e(E) := \left(\pi^{-1}\right)^* (u|_E) \in \mathrm{H}^n(B)$$

where $\cdot|_E \colon \mathrm{H}^n(E, E_0) \to \mathrm{H}^n(E)$ is the restriction map.

Proposition 3.1 (Properties of the Euler Class)**.** *Let $E \to B, E_1 \to B_1, E_2 \to B_2$ be oriented vector bundles.*

1. *If E admits a nonvanishing section, then $e(E) = 0$.*
2. *If $f \colon B_1 \to B_2$ is covered by a bundle map $\tilde{f} \colon E_1 \to E_2$ which is fiberwise an orientation preserving isomorphism, then*

   $$e(E_1) = f^* e(E_2)$$

3. *For the product bundle $E_1 \times E_2 \to B_1 \times B_2$,*

   $$e(E_1 \times E_2) = e(E_1) \times e(E_2)$$

   Let E_1, E_2 be two oriented vector bundles over B. Then

   $$e(E_1 \oplus E_2) = e(E_1) \smile e(E_2)$$

   where the orientation of $(E_1)_p \oplus (E_2)_p$ is given by an oriented basis of $(E_1)_p$ followed by an oriented basis of $(E_2)_p$.
4. *$e(E^-) = -e(E)$ where E^- is E with reversed orientation.*
5. *If n is odd, then $e(E) + e(E) = 0$.*

Theorem 3.2. *Let $E \to M$ a smooth oriented rank n real vector bundle over a closed oriented manifold M. Let ψ a section which intersects the zero section transversely and let $Z = \psi(M) \cap M$ where M is identified with the zero section of E. Then the Euler class of E is the Poincare dual of the fundamental class of Z:*

$$e(E) = [Z]^* = [\psi(B) \cap B]^* \in \mathrm{H}^n(B; \mathbb{Z})$$

Given a section ψ which intersects the zero section transversely, the zero set $Z = \psi^{-1}(0)$ is a submanifold of B and the derivative of ψ along the zero section defines an isomorphism of vector bundles

$$N_Z^B \cong E|_Z \tag{3.1}$$

This gives us an orientation of N_Z^B and thus an orientation of Z.

Proof. Let $u \in \mathrm{H}^n(E, E \setminus B; \mathbb{Z})$ the Thom class of E. Identify the normal bundle N_Z^B with an open tubular neighborhood N of Z in B. Let $u_Z \in \mathrm{H}^n(N, N \setminus Z; \mathbb{Z})$ be the Thom class of N_Z^B. The map $\psi|_N \colon (N, N \setminus Z) \to (E, E \setminus B)$ is homotopic through maps of pairs to the map $(N, N \setminus Z) \to (E|_Z, E|_Z \setminus Z)$ given by the isomorphism 3.1. Thus by naturality of the Thom class,

$$(\psi|_N)^* u = u_Z \in \mathrm{H}^n(N, N \setminus Z; \mathbb{Z})$$

We apply the composition $\mathrm{H}^n(N, N \setminus Z; \mathbb{Z}) \to \mathrm{H}^n(B, B \setminus Z; \mathbb{Z}) \to \mathrm{H}^n(B; \mathbb{Z})$ where the first is an excision isomorphism.

$$\psi^*(u|_E) = u_Z^B \in \mathrm{H}^n(B; \mathbb{Z}).$$

The left hand side is $e(E)$ by definition, the right hand side is $[Z]^*$ by lemma 2.4. □

Theorem 3.3. *Let M be a compact oriented n-manifold and $e(M)$ the Euler class of its tangent bundle. Then the Euler characteristic of M is*

$$\chi(M) = \langle e(M), [M] \rangle$$

Proof (Sketch). Let V be a vector field which intersects the zero section transversely. By the identity $[M] \frown e(M) = [Z]$, we have

$$\begin{aligned}\langle e(M), [M] \rangle &= \langle 1, [M] \frown e(M) \rangle = \langle 1, [Z] \rangle \\ &= \langle 1, [V(B) \cap B] = \sum_p I(V, M, p)\end{aligned}$$

where the sum is over all p with $V(p) = 0$ and $I(V, M, p)$ is the sign of intersection of V and the zero section at p.

We give two more characterizations of the intersection sign. The transversality condition is equivalent to the nonvanishing of the determinant of the map $\nabla V_p \colon T_p X \to T_p X$ (one can choose any connection here). Such a zero p of V is called nondegenerate. One can check that $I(V, M, p) = \deg(V, p) := \mathrm{sgn}(\det(\nabla V_p))$.

Using local coordinates around a zero p of V we can regard V as a map from $\mathbb{R}^n$ to $\mathbb{R}^n$ with $V(0) = 0$. Thus V gives a map $S^{n-1} \to \mathbb{R}^n \setminus \{0\} \cong S^{n-1}$. One checks that $\deg(V, p)$ is equal to the degree of this map if p is nondegenerate so we can take this as a general definition of $\deg(V, p)$. The theorem then follows from the following theorem. □

Theorem 3.4 (Poincaré-Hopf)**.** *If X is a closed smooth manifold and V a smooth vector field with isolated zeros, then*

$$\sum_{V(p)=0} \deg(V,p) = \chi(X)$$

We will sketch the proof of the theorem in the case that all zeros of V are nondegenerate and X is oriented, using the Lefschetz fixed point theorem.

Let $f\colon X \to X$ with only isolated fixed points, we define the Lefschetz sign $\epsilon(p)$ of a fixed point p of f as follows. Choosing local coordinates around p, we can regard f as a map $S^{n-1} \to \mathbb{R}^n \setminus \{0\} \cong S^{n-1}$ and we take $\epsilon(p)$ to be the degree of that map. If f is smooth, we call a fixed point nondegenerate if the derivative $Id - df_p$ is invertible. Note that this is equivalent to the condition that the intersection of the graph $\Gamma(f)$ of f and the diagonal $\Delta \subseteq M \times M$ is transverse in $M \times M$. In this case the intersection degree of $\Gamma(f)$ and Δ at p is $I(\Gamma(f), \Delta, p) = \operatorname{sgn}(\det(1 - df_p))$ and one can check that $\epsilon(p) = \operatorname{sgn}(\det(1 - df_p))$. Note that by these definitions, $[\Gamma(f)] \cdot [\Delta] = \sum_{f(p)=p} \epsilon(p)$. The Lefschetz fixed point theorem states

Theorem 3.5 (Lefschetz fixed point theorem)**.** *If X is a closed oriented smooth manifold and $f\colon X \to X$, then*

$$L(f) := [\Gamma(f)] \cdot [\Delta] = \sum_{i=1}^{k} (-1)^i \operatorname{tr}\left(f_*\colon\ \mathrm{H}_i(X, \mathbb{Q}) \to \mathrm{H}_i(X, \mathbb{Q})\right)$$

In the case that f is homotopic to the identity, we obtain

$$[\Gamma(f)] \cdot [\Delta] = \sum_{i=1}^{k} (-1)^k \dim(\mathrm{H}_i(X, \mathbb{Q})) = \chi(X) = [\Delta] \cdot [\Delta]$$

Proof of 3.4(Sketch). Assume X is oriented and all fixed points of V are nondegenerate. In this case for small t, the map $f = \exp(tV)$ is a diffeomorphism of M and all nondegenerate fixed points of f correspond to nondegenerate zeroes of V. One can check that $\deg(V,p) = \epsilon(p)$. Thus since f is homotopic to the identity,

$$\sum_{V(p)=0} \deg(V,p) = \sum_{f(p)=p} \epsilon(p) = \sum_{i=1}^{k} \dim \mathrm{H}_i(M; \mathbb{Q}) = \chi(M)$$

□

One can get rid of the nondegeneracy assumption by showing that the left hand side of the equation does not depend on V.

Proof of 3.5 (Sketch). We will use the following rules of calculation without proof: If $\alpha, \beta, \gamma, \delta \in \mathrm{H}_*(X)$ with $|\alpha| + |\beta| = |\gamma| + |\delta| = \dim X$

$$(\alpha \times \beta) \cdot (\gamma \times \delta) = \begin{cases} (-1)^{|\beta|}(\alpha \cdot \gamma)(\beta \cdot \delta) & \text{if} |\beta| = |\gamma| \\ 0 & \text{otherwise} \end{cases}$$

$$[\Gamma(f)] \cdot (\alpha \times \beta) = (-1)^{|\alpha|} f_* \alpha \cdot \beta \in \mathbb{Z}$$

Note that since the cup product is a perfect pairing on a closed connected orientable manifold, the intersection product is also a perfect pairing. Fix a basis $e_k \in \mathrm{H}_*(X;\mathbb{Q})$ and take the dual basis $e'_k \in \mathrm{H}^*(X;\mathbb{Q})$ with respect to the intersection product, i.e. such that $e_i \cdot e'_j = \delta_{ij}$.

The homology cross product is an isomorphism $\mathrm{H}_*(X \times X;\mathbb{Q}) \cong \mathrm{H}(X;\mathbb{Q}) \otimes \mathrm{H}(X;\mathbb{Q})$. Thus $\{e_i \times e'_j\}$ is a basis of $\mathrm{H}_*(X \times X;\mathbb{Q})$. We will first prove the equation $[\Delta] = \sum_k e_k \times e'_k$ by verifying that the identity holds if we take its intersection product the basis elements $e'_i \times e_j$ when $|e'_i| + |e_j| = n$. Taking $f = id_X$ in the computation rule above we can compute

$$\begin{aligned}
\left(\sum_k e_k \times e'_k\right) \cdot (e'_i \times e_j) &= \sum_{k:|e_k|=|e'_i|} (-1)^{|e'_i|}(e_k \cdot e'_i)(e'_k \cdot e_j) \\
&= (-1)^{|e'_i|} e'_i \cdot e_j \\
&= [\Delta](e'_i \times e_j)
\end{aligned}$$

Now we can prove the theorem.

$$\begin{aligned}
[\Gamma(f)] \cdot [\Delta] &= [\Gamma(f)] \cdot \sum_k e_k \times e'_k \\
&= \sum_k (-1)^{|e_k|} f_* e_k \cdot e'_k \\
&= \sum_i (-1)^i \operatorname{tr}(f_*\colon\ \mathrm{H}_i(X;\mathbb{Q}) \to \mathrm{H}_i(X;\mathbb{Q}))
\end{aligned}$$

$\square$