recall standard Sobolev estimates (fns. on a domain in $\mathbb{R}^n$)

(1) $W^{k+d,p} \hookrightarrow C^d$ if $kp > n$

(2) $W^{k,p} \hookrightarrow W^{m,q}$ if $k \geq m$, $k - \frac{n}{p} \geq m - \frac{n}{q}$, $q \geq p$

(3) $W^{k,p} \times W^{m,q} \longrightarrow W^{m,q} : (f,g) \longmapsto fg$ if $k \geq m$, $k - \frac{n}{p} \geq m - \frac{n}{q}$.
$kp > n$.

(4) $C^k \times W^{k,p} \longrightarrow W^{k,p} : (f,g) \longmapsto f \circ g$ contin. if $kp > n$.

elliptic estimate of $\bar{\partial}$:

(5) $\|u\|_{W^{k,p}} \leq c \|\bar{\partial} u\|_{W^{k-1,p}} \quad \forall\, u \in W_0^{k,p}$

<u>linear regularity thm</u>: Fix $k \in \mathbb{N}$, $1 < p < \infty$, $m \geq k$.

(1) $\bar\partial u = f$ for $u \in W^{k,p}$, $f \in W^{m,p}$ on $\mathbb{D}$ $\Rightarrow$ $u \in W^{m+1,p}_{loc}(\mathring{\mathbb{D}})$

(i.e. $W^{m+1,p}$ on all cpct subsets of $\mathring{\mathbb{D}}$).

(<u>cor</u>: $f \in C^\infty \Rightarrow u \in \bigcap_{m \in \mathbb{N}} W^{m,p}_{loc} = C^\infty$.)

(2) Seqs. $u_\nu \in W^{k,p}$, $f_\nu \in W^{m,p}$, $\bar\partial u_\nu = f_\nu$.

(a) $\|u_\nu\|_{W^{k,p}}$, $\|f_\nu\|_{W^{m,p}}$ unif bdd on $\mathbb{D}$ $\Rightarrow$ $\|u_\nu\|_{W^{m+1,p}}$ bdd on all cpct subset of $\mathring{\mathbb{D}}$.

(<u>cor</u> (via Arzelà-Ascoli):

$f_\nu \xrightarrow{C^\infty} f$ & $\|u_\nu\|_{W^{1,p}}$ unif bdd $\Rightarrow \exists$ C^∞_{loc}-conv. subseq.)

(b) $u_\nu \xrightarrow{W^{k,p}} u$ & $f_\nu \xrightarrow{W^{m,p}} f \Rightarrow u_\nu \xrightarrow{W^{m+1,p}_{loc}} u$.

(<u>cor</u>: all reasonable Sobolev-type topologies on sol. spaces $\cong C^\infty$-top.)

<u>EX</u>: Same result holds w/ $\bar\partial$ replaced with $\bar\partial + A$ for $A \in C^m$ or $\bar\partial + A_\nu$ with $A_\nu \xrightarrow{C^m} A$.

pf of (2a), assuming (1): Suff to consider $k = m$: $\|u_\nu\|_{W^{k,p}}, \|f_\nu\|_{W^{k,p}} \le C$.

Fix $r \in (0,1)$ & $\beta \in C_0^\infty(\mathbb{D}, [0,1])$ w/ $\beta|_{\mathbb{D}_r} \equiv 1$. (1) $\Rightarrow$ $u_\nu \in W^{k+1,p}_{loc}$

$\Rightarrow$ for $\partial_1 = \partial_s, \partial_2 = \partial_t$, $\beta \partial_j u_\nu \in W_0^{k,p}$ for $j = 1,2$.

$$\|\partial_j u_\nu\|_{W^{k,p}(\mathbb{D}_r)} \le \|\beta \partial_j u_\nu\|_{W^{k,p}(\mathbb{D})} \overset{5}{\le} c \|\bar\partial(\beta \partial_j u_\nu)\|_{W^{k-1,p}}$$

$$\le c \|\underbrace{(\bar\partial\beta)\partial_j u_\nu}_{\text{bdd}}\|_{W^{k-1,p}} + c\|\underbrace{\beta \partial_j f_\nu}_{\text{bdd}}\|_{W^{k-1,p}}$$

∎

pf of (2b) similar.

pf of (1): Replace $\partial_j u_\nu$ w/ difference quotients $D_j^h u(z) := \dfrac{u(z+he_j) - u(z)}{h}$

for $h \in \mathbb{R}\setminus\{0\}$ small, $e_1 = \partial_s$, $e_2 = \partial_t$.

$u \in W^{k,p} \Rightarrow D_j^h u \in W^{k,p}$, $\partial_j u \in W^{k-1,p} \Rightarrow \|D_j^h u\|_{W^{k-1,p}}$ bdd as $h \to 0$,

sim. $\|D_j^h f\|_{W^{m-1,p}}$ bdd. Same argument as above $\rightsquigarrow$

$\|D_j^h u\|_{W^{k,p}(\mathbb{D}_r)}$ bdd as $h \to 0$. $\Rightarrow$ $\partial_j u \in W^{k,p}(\mathbb{D}_r)$.

(e.g. if $u \in L^p$ & $\|D_j^h u\|_{L^p}$ bdd as $h \to 0$, then $\forall$ seqs. $h_\nu \to 0$,

$1 < p < \infty$, Banach-Alaoglu thm $\Rightarrow$ $D_j^{h_\nu} u$ has a weakly L^p-conv. subseq.

i.e. $\forall$ $g \in L^q$ w/ $\frac{1}{p} + \frac{1}{q} = 1$, $\int (D_j^{h_\nu} u)\, g \longrightarrow \int \underbrace{(\text{limit})}_{\in L^p} g$.

<u>EX</u>: limit = weak deriv. $\partial_j u$.)

∎

lemma: Weak sols. $u \in L^1_{loc}$ to $\bar{\partial} u = 0$ are smooth.

"pf": Re u, Im u are "weakly" harmonic fns., by mollification, can approximate them w/ C^∞ harmonic fns conv. in L^1_{loc}.

Can characterize harmonic fns via mean value property:

$f(z) = \frac{1}{\pi r^2} \int_{D_r(z)} f(\zeta)\, dm(\zeta) \Rightarrow L^1$-limits also satisfy MVP. $\boxtimes$

cor: For $1 < p < \infty$ & $A \in C^\infty$, weak sols of class L^p to $(\bar{\partial} + A) u = 0$ are smooth.

Pf: Recall $\exists$ right-inverse $T : L^p \to W^{1,p}$ of $\bar{\partial}$. $u \in L^p$ &

$\bar{\partial} u = \underbrace{-Au}_{=: -f} \in L^p$. $Tf \in W^{1,p}$ sats. $\bar{\partial}(Tf) = f = \bar{\partial} u \Rightarrow$

$u - Tf \in L^p$ is a weak sol. to $\bar{\partial}(u - Tf) = 0 \Rightarrow u - Tf \in C^\infty \Rightarrow u \in W^{1,p}_{loc}$.

Now shrink disk s.t. $u \in W^{1,p}$, then $\bar{\partial} u = -Au \in W^{1,p} \Rightarrow u \in W^{2,p}_{loc}$, continue... $\boxtimes$

nonlinear regularity thm: If $kp > 2$ & $J : \mathring{\mathbb{D}} \times \mathbb{C}^n \longrightarrow \mathcal{J}(\mathbb{C}^n) := \left\{ \begin{array}{l} \mathbb{R}\text{-lin. maps } K : \mathbb{C}^n \to \mathbb{C}^n \\ \text{w/ } K^2 = -\mathrm{Id} \end{array} \right\}$

is of class C^m (or $J_\nu \xrightarrow{C^m_{loc}} J$), then statements (1), (2a), (2b)

are also true for sol. of the nonlinear eqn. $\partial_s u(z) + J(z, u(z)) \partial_t u(z) = f(z)$.

idea: Given $z_0 \in \mathring{\mathbb{D}}$, can change coords on $\mathbb{C}^n$ s.t. WLOG $u(z_0) = 0$, $J(z_0, 0) = i$.

rescaling trick: For $\varepsilon > 0$ & a suitable const. $\alpha \in (0,1)$, replace

$$u \rightsquigarrow \hat{u}(z) := \frac{u(z_0 + \varepsilon z)}{\varepsilon^\alpha} \qquad f \rightsquigarrow \hat{f}(z) := \varepsilon^{1-\alpha} f(z_0 + \varepsilon z).$$

$$J \rightsquigarrow \hat{J}(z, x) := J(z_0 + \varepsilon z, \varepsilon^\alpha x)$$

Now $\partial_s u + J(z,u) \partial_t u = f \iff \partial_s \hat{u} + \hat{J}(z, \hat{u}) \partial_t \hat{u} = \hat{f}$,

but for ε small, can assume $\| \hat{J} - i \|_{C^m(\mathbb{D} \times \mathbb{D}^{2n})}$ small.

Now in estimating $\| \beta \, \partial_j \hat{u}_\nu \|_{W^{k,p}}$, $\exists$ additional terms such as

$$\| \underbrace{[\hat{J}_\nu(z, \hat{u}_\nu) - i]}_{W^{k,p}} \underbrace{\partial_t (\beta \partial_j \hat{u}_\nu)}_{W^{k-1,p}} \|_{W^{k-1,p}} \overset{(3)}{\leq} c \| \hat{J}_\nu(z, \hat{u}_\nu) - i \|_{W^{k,p}} \cdot \| \partial_t (\beta \partial_j \hat{u}_\nu) \|_{W^{k-1,p}}$$

$$\leq c \| \hat{J}_\nu(z, \hat{u}_\nu) - i \|_{W^{k,p}} \cdot \| \beta \partial_j \hat{u}_\nu \|_{W^{k,p}}$$

If can assume $\hat{u}_\nu$ $W^{k,p}$-small for $\varepsilon > 0$ suff. small, then $\hat{J} - i$ C^k-small

$\overset{(4)}{\Longrightarrow}$ can assume $\| \hat{J}_\nu(z, \hat{u}_\nu) - i \|_{W^{k,p}}$ arb. small, e.g. $< \frac{1}{2c}$.

$\Rightarrow$ can move $\frac{1}{2} \| \beta \partial_j \hat{u}_\nu \|_{W^{k,p}}$ to the LHS.

Lemma (cor. of Sobolev emb. thm): If $kp > n$ & $\alpha \in (0,1)$ s.t.

$\alpha \leq k - \frac{n}{p}$, then for $f_\varepsilon(x) := f(x_0 + \varepsilon x)$, $\exists$ $C > 0$ (indep. of f)

s.t. $\| f_\varepsilon - f_\varepsilon(0) \|_{W^{k,p}(\mathbb{D}^n)} \leq C \varepsilon^\alpha \| f - f(x_0) \|_{W^{k,p}(\mathbb{D}^n)}$.

ex: $f \in W^{k,p}$, β a multi-index of order k, then $\partial^\beta f_\varepsilon(x) = \varepsilon^k \partial^\beta f(x_0 + \varepsilon x)$

$$\Rightarrow \| \partial^\beta f_\varepsilon \|^p_{L^p} = \int_{\mathbb{D}^n} | \partial^\beta f_\varepsilon(x) |^p = \varepsilon^{kp} \int_{\mathbb{D}^n} | \partial^\beta f(x_0 + \varepsilon x) |^p = \varepsilon^{kp-n} \int_{\mathbb{D}^n_\varepsilon(x_0)} | \partial^\beta f(x) |^p$$

$$\leq \varepsilon^{kp-n} \| \partial^\beta f \|^p_{L^p}. \qquad \| \partial^\beta f_\varepsilon \|_{L^p} \leq \varepsilon^{k - \frac{n}{p}} \| \partial^\beta f \|_{L^p}.$$

isolated intersections

thm: $u, v : (\mathbb{D}, i) \to (\mathbb{C}^n, J)$ J-hol. w/ $u(0) = v(0)$, then $\exists$ nbhds $\mathcal{O}, \mathcal{O}' \subseteq \mathbb{D}$ of 0 s.t. either $u(\mathcal{O}) = v(\mathcal{O}')$

or $\left(u(\mathcal{O}) \cap v(\mathcal{O}' \setminus \{0\})\right) \cup \left(u(\mathcal{O} \setminus \{0\}) \cap v(\mathcal{O}')\right) = \emptyset$.

pf in case $du(0) \neq 0$: Choose coords on $\mathbb{C}^n$ s.t. WLOG

$u(z) = (z, 0) \in \mathbb{C} \times \mathbb{C}^{n-1}$ & $J(z, 0) = i$ $\forall$ $z \in \mathbb{D}$.

Then for $(z, w) \in \mathbb{C} \times \mathbb{C}^{n-1}$, $J(z, w) = i + \int_0^1 \frac{d}{d\tau} J(z, \tau w)\, d\tau$

$$= i + \underbrace{\left(\int_0^1 D_2 J(z, \tau w)\, d\tau \right)}_{B(z,w) \,:\, \mathbb{C}^{n-1} \xrightarrow{\mathbb{R}\text{-lin}} \operatorname{End}_{\mathbb{R}}(\mathbb{C}^n)} w$$

Write $v(z) = (\varphi(z), f(z)) \in \mathbb{C} \times \mathbb{C}^{n-1}$. Then

$$0 = \partial_s v + J(\varphi, f)\, \partial_t v = \partial_s v + i\, \partial_t v + \left[B(\varphi, f) f \right] \partial_t v$$

Let $\pi : \mathbb{C} \times \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$ proj. Then f satisfies $\partial_s f + i \partial_t f + Af = 0$

where $A : \mathbb{D} \to \operatorname{End}_{\mathbb{R}}(\mathbb{C}^{n-1})$ def'd by

$$A(z) w := \pi \left[B(\varphi(z), f(z)) w \right] \partial_t v. \qquad (\bar{\partial} + A) f = 0$$

By similarity princ., either $f \equiv 0$ near 0 or the origin is an isolated zero of f. $\boxtimes$