asymptotic operators & spectral flow

motivation (Morse homology): (M^n, g) Riem. mfd. $f : M \longrightarrow \mathbb{R}$ Morse

$x_\pm \in \mathrm{Crit}(f)$

$$\mathcal{M}(x_-, x_+) := \left\{ u \in C^\infty(\mathbb{R}, M) \;\middle|\; \dot u(s) + \nabla f(u(s)) = 0, \; \lim_{s \to \pm\infty} u(s) = x_\pm \right\}$$

linearize: $\{u_\rho : \mathbb{R} \longrightarrow M\}_{\rho \in (-\varepsilon, \varepsilon)}$, $\quad u_0 = u$, $\quad \partial_\rho u_\rho|_{\rho = 0} = \eta \in \Gamma(u^* TM)$,

$$0 = \nabla_\rho \left[\partial_s u_\rho + \nabla f(u_\rho) \right]\Big|_{\rho = 0} = \nabla_s \eta + \nabla_\eta \nabla f =: D_u \eta \in \Gamma(u^* TM)$$

(assuming ∇ symmetric)

choose triv. of $u^* TM$: D_u becomes $C^\infty(\mathbb{R}, \mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}, \mathbb{R}^n)$:

$$D_u \eta = \dot\eta + A(s)\eta(s) \quad \text{where } A(s) \in \mathbb{R}^{n \times n} \quad \text{s.t. } \lim_{s \to \pm\infty} A(s) = A_\pm = \nabla^2 f(x_\pm)$$

symmetric (Hessian)

$A_\pm$ nonsingular $\Longleftrightarrow$ $x_\pm$ Morse.

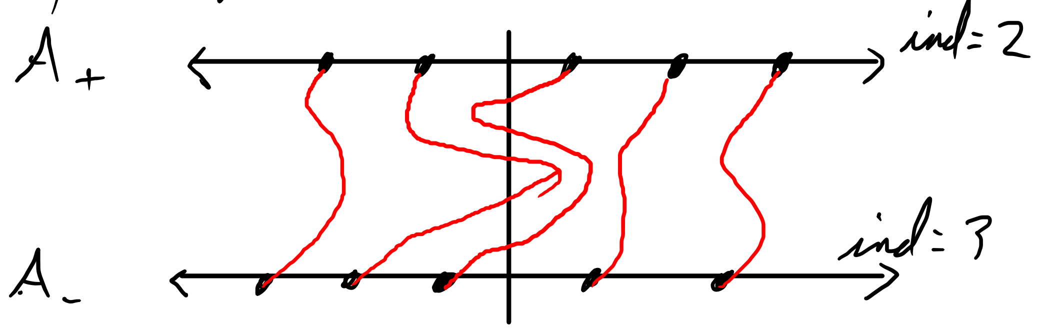
Thm (see Schwarz "Morse hom."): Operators of the form

$$D : W^{k,p}(\mathbb{R}, \mathbb{R}^n) \longrightarrow W^{k-1,p}(\mathbb{R}, \mathbb{R}^n) : \eta \longmapsto \dot\eta + A(s)\eta(s) \qquad \begin{pmatrix} k \in \mathbb{N}, \\ 1 < p < \infty \end{pmatrix}$$

w/ $\lim_{s \to \pm\infty} A(s) = A_\pm$ symmetric & nonsingular

are Fredholm & $\mathrm{ind}(D) = \dim E^-(A_-) - \dim E^-(A_+) =$ "spectral flow from A_- to A_+"

$(E^-(\cdot) :=$ neg. def. subspace$)$



to prove: In Floer-type theories,

$\dim E^-(A_\pm) = \infty$, but "$\dim E^-(A_-) - \dim E^-(A_+)$"

is a well-def'd integer, expressed in terms of spectral flow.

Hessian in contact geometry

$(M^{2n-1}, \xi = \ker\alpha)$ ctct mfd, R_α = Reeb vector fld, cpx str. $J : \xi \to \xi$
compat. w/ $d\alpha|_\xi$, $\pi_\xi : TM = \mathbb{R}R_\alpha \oplus \xi \to \xi$ proj.

Defn <u>ctct action fncl</u> $\mathcal{A} : C^\infty(S^1, M) \to \mathbb{R} : \gamma \mapsto \int_{S^1} \gamma^*\alpha$.

<u>EX</u>: For $\eta \in \Gamma(\gamma^*\xi)$, $d\mathcal{A}(\gamma)\eta = \int_{S^1} d\alpha(\eta(t), \dot\gamma(t))\, dt = \int_{S^1} d\alpha(\eta, \pi_\xi \dot\gamma)\, dt$

$$= \int_{S^1} d\alpha(J\eta, J\pi_\xi\dot\gamma)\, dt = \int_{S^1} d\alpha(-J\pi_\xi\dot\gamma, J\eta)\, dt = \langle \underbrace{-J\pi_\xi\dot\gamma}_{=: \nabla\mathcal{A}(\gamma)}, \eta \rangle_{L^2}$$

where $\langle \eta_1(t), \eta_2(t) \rangle_{L^2} := \int_{S^1} d\alpha(\eta_1, J\eta_2)\, dt$ on $\Gamma(\gamma^*\xi)$.

$\Rightarrow$ $\gamma \in \mathrm{Crit}(\mathcal{A})$ iff $\dot\gamma \propto R_\alpha$. Choose a parametrization of γ s.t.
$\alpha(\dot\gamma) =: T > 0$ const. (= period) & differentiate $\nabla\mathcal{A}$ at γ in direction $\Gamma(\gamma^*\xi)$:

$$\nabla^2\mathcal{A}(\gamma)\eta := \pi_\xi \nabla_\rho [\nabla\mathcal{A}(\gamma_\rho)]\big|_{\rho=0} = \pi_\xi \nabla_\rho \left(-J(\gamma_\rho)\, \pi_\xi\, \partial_t\gamma_\rho\right)\big|_{\rho=0}$$

$$= -\pi_\xi \nabla_\rho \left(J(\gamma_\rho)\left(\partial_t\gamma_\rho - \alpha(\partial_t\gamma_\rho) R_\alpha(\gamma_\rho)\right)\right)\big|_{\rho=0}$$

$$= \boxed{-J(\nabla_t\eta - T\nabla_\eta R_\alpha) =: \mathbf{A}_\gamma \eta}$$

<u>EX</u>: $\mathbf{A}_\gamma\eta \in \Gamma(\gamma^*\xi)$ & is indep. of choice of (symmetric) connection on M.

$\mathbf{A}_\gamma$ = the "<u>asymptotic operator</u> of γ"

EX: (prelim. remark: $\mathcal{L}_{R_\alpha} \alpha = d \underbrace{\iota_{R_\alpha} \alpha}_{=1} + \underbrace{\iota_{R_\alpha} d\alpha}_{=0} = 0$

$$\mathcal{L}_{R_\alpha} d\alpha = d \underbrace{\iota_{R_\alpha} d\alpha}_{0} + \iota_{R_\alpha} \underbrace{d^2 \alpha}_{=0} = 0$$

$\Rightarrow$ flow of R_α preserves ξ & its sympl. bndl str. $d\alpha|_\xi$.)

$A_\gamma = -J \nabla_t^\alpha$ where $\nabla^\alpha :=$ the ! symplectic connection on $(\gamma^* \xi, d\alpha|_\xi)$

s.t. parallel transport = linearized flow of R_α along γ.

<u>defn</u>: γ is nondegenerate if $\ker A_\gamma = \{0\}$.

<u>defn</u>: <u>Hermitian vector bndl</u> $:= (E, J, \omega)$, $J \in \Gamma(\mathrm{End}(E))$ s.t. $J^2 = -\mathrm{Id}$,

$\omega \in \Gamma(\Lambda^2 E^*)$ s.t. $g := \omega(\cdot, J\cdot)$ is a bndl metric.

$(\Rightarrow \langle\, ,\, \rangle := g + i\omega$ is a Hermitian bndl metric.)

For (E, J, ω) over S^1, an <u>asymptotic op.</u> $A : \Gamma(E) \longrightarrow \Gamma(E)$ is any op. of form $A = -J \nabla_t$ for ∇ a symplectic connection.

<u>EX</u>: $A = -J \nabla_t$ is symmetric w.r.t. $\langle \eta_1, \eta_2 \rangle_{L^2} = \int_{S^1} \omega(\eta_1(t), J \eta_2(t))\, dt$.

<u>EX</u>: In any <u>unitary</u> <u>triv.</u> of (E, J, ω) (i.e. s.t. fibers $= \underset{\mathbb{R}^{2n}}{\overset{\shortparallel}{\mathbb{C}^n}}$, $J = i$, $\omega = \omega_{std}$) A becomes $C^\infty(S^1, \mathbb{R}^{2n}) \longrightarrow C^\infty(S^1, \mathbb{R}^{2n})$ of the form $A = -i \partial_t - S(t)$ for some smooth loop $S : S^1 \longrightarrow \mathrm{End}_{\mathbb{R}}^{sym}(\mathbb{R}^{2n})$.

<u>Lemma</u>: $A := -i\partial_t - S(t) : \overset{W^{1,2}}{\shortparallel}H^1(S^1, \mathbb{C}^n) \longrightarrow L^2(S^1, \mathbb{C}^n)$ is Fredholm w/ index 0.

<u>pf</u>: $H^1 \xhookrightarrow{\text{cpct}} L^2 \xrightarrow{\eta \mapsto S\eta} L^2$ is a cpct op. $\Rightarrow$ WLOG, can restrict attention to $-i\partial_t$ = isomorphism $\circ\, \partial_t$ $\Rightarrow$ just consider ∂_t.

$\underbrace{\ker \partial_t = \{\text{const. fns. } S^1 \to \mathbb{C}^n\}}_{\dim_{\mathbb{R}} = 2n}$, $\quad \operatorname{im} \partial_t = \underbrace{\left\{ f \in L^2(S^1, \mathbb{R}^{2n}) \,\middle|\, \int_{S^1} f(t)\, dt = 0 \right\}}_{\operatorname{codim}_{\mathbb{R}} = 2n}$ $\boxtimes$

View $A = -J\nabla_t : \Gamma(E) \longrightarrow \Gamma(E)$ as unbdd op. on $L^2(E)$ with dense domain $H^1(E) \subseteq L^2(E)$.

<u>spectrum</u>: $\sigma(A) := \left\{ \lambda \in \mathbb{C} \;\middle|\; A^{\mathbb{C}} - \lambda : H^1(E^{\mathbb{C}}) \longrightarrow L^2(E^{\mathbb{C}}) \text{ has no bdd inverse } L^2 \longrightarrow H^1 \right\}$.

complexification

<u>cor</u> (since $A^{\mathbb{C}} - \lambda$ also has Fredholm index 0): $\sigma(A)$ consists only of eigenvalues w/ finite multiplicity. ☒

<u>symmetric</u> $\Longrightarrow \sigma(A) \subseteq \mathbb{R}$.

spectral flow thm: Consider a smooth 1-param. fam. trivialized asymp. ops. $\{A_s = -i\partial_t - S_s(t) : H^1(S^1,\mathbb{C}^n) \longrightarrow L^2(S^1,\mathbb{C}^n)\}_{s\in[-1,1]}$.

$\exists$ contin. fn. $\{\lambda_j : [-1,1] \longrightarrow \mathbb{R}\}_{j\in\mathbb{Z}}$ s.t. $\forall\ s\in[-1,1]$,

$\sigma(A_s) = \{\lambda_j(s) \mid j\in\mathbb{Z}\}$ counted w/ multiplicity,

& if $A_\pm := A_{\pm 1}$ both have trivial kernel, then

$$\left.\begin{aligned} \mu^{spec}(A_-,A_+) := \#\{j\in\mathbb{Z} \mid \lambda_j(-1) < 0 < \lambda_j(1)\} - \\ \#\{j\in\mathbb{Z} \mid \lambda_j(-1) > 0 > \lambda_j(1)\} \in \mathbb{Z} \end{aligned}\right\} \text{spectral flow}$$

deps. only on $A_\pm$, not on $\{A_s\}_{s\in[-1,1]}$.

main lemma (tomorrow): $\forall\ k \in \mathbb{N}$,

$$\mathrm{Fred}_{\mathbb{R}}^{\mathrm{sym},k} := \{ T \in \underbrace{\mathcal{L}_{\mathbb{R}}^{\mathrm{sym}}(H^1, L^2)}_{\substack{\text{bdd, } \mathbb{R}\text{-linear} \\ \text{symmetric}}} \mid \dim \ker T = \operatorname{codim} \operatorname{im} T = k \}$$

is a smooth finite-codimensional submfd of $\mathcal{L}_{\mathbb{R}}^{\mathrm{sym}}(H^1, L^2)$ with

codim $= \dfrac{k(k+1)}{2}$, & for $k = 1$ it has a canonical co-orientation.

Moreover, $\{ (T, \eta) \in \mathrm{Fred}_{\mathbb{R}}^{\mathrm{sym},k} \times H^1 \mid T\eta = 0 \}$ is a smooth vec. bndl.

Can then defn. $\mu^{\mathrm{spec}}(A_-, A_+) :=$ signed count of intersections of the path

$[-1,1] \longrightarrow \mathcal{L}_{\mathbb{R}}^{\mathrm{sym}}(H^1, L^2) : s \longmapsto A_s$ with $\mathrm{Fred}_{\mathbb{R}}^{\mathrm{sym},1}$,

& reinterpret in terms of e-vals.