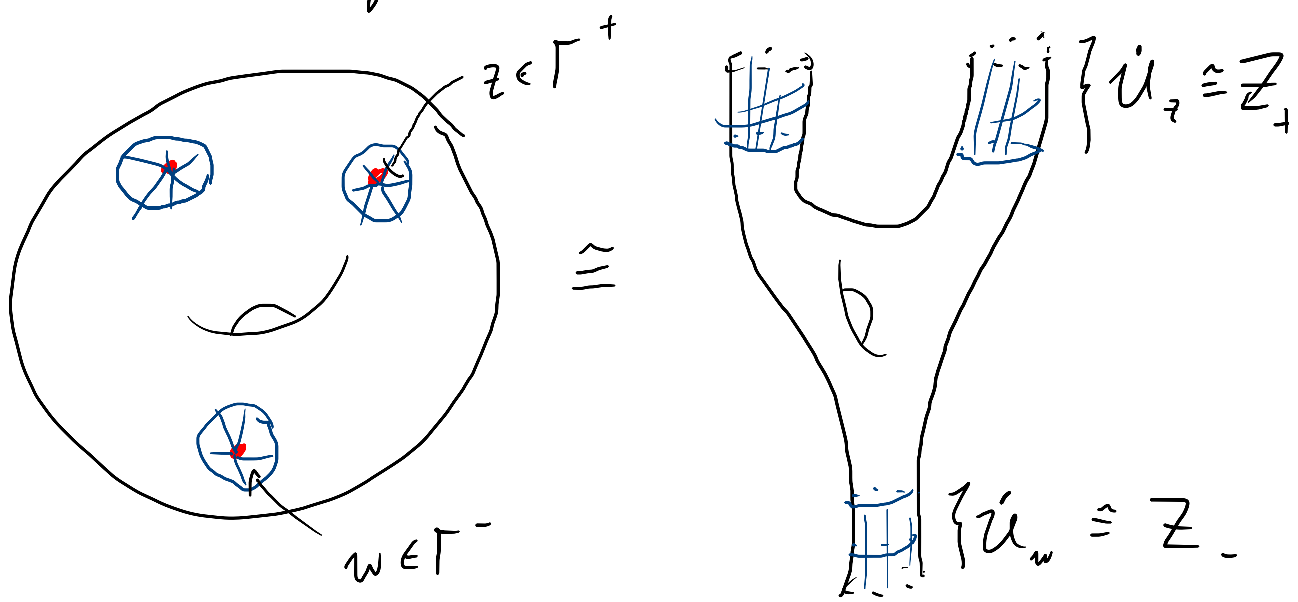
The linearized CR-op. in SFT

(Σ, j) closed Riem. surf., $\Gamma = \Gamma^+ \amalg \Gamma^- \subseteq \Sigma$ finite

$\dot{\Sigma} := \Sigma \setminus \Gamma$ = "R.S. w/ cylindrical ends"



$Z_+ := [0,\infty) \times S^1$, $\quad Z_- := (-\infty, 0] \times S^1$

$$\begin{array}{cccc} Z_+ & (s,t) & Z_- & (s,t) \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \mathbb{D}\setminus\{0\} & e^{-2\pi(s+it)} & \mathbb{D}\setminus\{0\} & e^{2\pi(s+it)} \end{array}$$

An asymptotically Hermitian V.B. over $(\dot{\Sigma}, j)$ consists of:

(i) a cpx V.B. $(E, J) \longrightarrow \dot{\Sigma}$

(ii) Hermitian vec. bundles $(E_z, J_z, \omega_z) \longrightarrow S^1$ associated to each $z \in \Gamma^\pm$

(iii) cpx VB-isos. $\begin{array}{ccc} E|_{\mathcal{U}_z} & \longrightarrow & \pi^* E_z \\ \downarrow & & \downarrow \\ \mathcal{U}_z & \xrightarrow{\sim} & Z_\pm \end{array}$ for $\pi : Z_\pm \longrightarrow S^1$ proj.

Unitary trivs. τ of $\{E_z\}_{z\in\Gamma}$ $\rightsquigarrow$ "asymptotic trivialization" of E on cyl. ends.

$W^{k,p}(E) := \{ \eta \in W^{k,p}_{loc}(E) \mid \eta|_{\mathcal{U}_z} \in W^{k,p}(Z_\pm, \mathbb{C}^n) \ \forall\, z \in \Gamma^\pm$ w.r.t. asymp. trivs.$\}$

<u>defn</u>: Cauchy-Riemann type op. of class C^m on E: $D : C^{m+1}(E) \longrightarrow C^m(F)$
s.t. locally in coords. & trivs. $D = \bar\partial + S$ where S is of class C^m.

D is C^m-asymptotic to an asymp. op. A_z on (E_z, J_z, ω_z) at $z \in \Gamma^\pm$ if in some asymp. triv., $A_z = -i\,\partial_t - S_\infty(t)$, $\quad D = \bar\partial + S(s,t)$ on $\mathcal{U}_z$

& $\|S - S_\infty\|_{C^m(Z^R_\pm)} \xrightarrow{R\to\infty} 0$ $\quad \left(\underline{\text{defn}}: \begin{array}{l} Z^R_+ := [R,\infty) \times S^1 \\ Z^R_- := (-\infty, -R] \times S^1 \end{array} \right)$

main thm: Assume D = a CR-type op. of class C^m $(0 \le m \le \infty)$

C^m-asymptotic to a nondeg. asymp. op. A_z at each $z \in \Gamma$.

Then $D : W^{k,p}(E) \longrightarrow W^{k-1,p}(F)$ is Fredholm $\forall\ k \in \{1, \dots, m+1\}$,

$1 < p < \infty$.

$$F = \overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, E)$$

Moreover, ind D & ker D are indep. of k, p.

(next week: $\mathrm{ind}(D) = \mathrm{rk}_{\mathbb{C}}(E) \cdot \chi(\dot{\Sigma}) + 2\underbrace{c_1^\tau(E)}_{\text{relative 1st Chern \# of } E \text{ w.r.t. the asymp. trivs. } \tau} + \sum_{z \in \Gamma^+} \mu_{CZ}^\tau(A_z) - \sum_{z \in \Gamma^-} \mu_{CZ}^\tau(A_z)$)

main idea: for $D = \bar\partial + S$ on $\mathbb{D}$ w/ $S \in C^m$, $1 \le k \le m+1$,

$$\|u\|_{W^{k,p}(\mathbb{D}_r)} \le c\|Du\|_{W^{k-1,p}(\mathbb{D})} + c\|u\|_{W^{k-1,p}(\mathbb{D})} \quad \forall u \in W^{k,p}(\mathbb{D}),\ 0<r<1.$$

(finite open covering) $\Longrightarrow$ Lemma 1: Given $\Sigma_0 \overset{\text{open}}{\subseteq} \bar\Sigma_0 \subseteq \Sigma_1 \overset{\text{open}}{\subseteq} \bar\Sigma_1 \overset{\text{cpt}}{\subseteq} \dot\Sigma$, $\exists\ c>0$ s.t.

$$\|\eta\|_{W^{k,p}(\Sigma_0)} \le c\|D\eta\|_{W^{k-1,p}(\Sigma_1)} + c\|\eta\|_{W^{k-1,p}(\Sigma_1)}.$$

If $\Gamma = \emptyset$, can take $\Sigma_0 = \Sigma_1 = \Sigma$, note $W^{k,p}(\Sigma) \hookrightarrow W^{k-1,p}(\Sigma)$ is a cpt op.

abstract lemma: X, Y, Z Banach spaces, $D \in \mathcal{L}(X,Y)$, $K \in \mathcal{L}(X,Z)$ a cpt op., & $\exists\ c>0$ s.t. $\|x\|_X \le c\|Dx\|_Y + c\|Kx\|_Z \quad \forall x \in X$.

Then $\dim \ker D < \infty$ & im D is closed (i.e. D is "semi-Fredholm").

pf: (1) $\dim\ker D < \infty \iff$ closed unit ball in $\ker D$ is cpt.

Spse $x_n \in \ker D$, $\|x_n\|_X \le 1$. $\Rightarrow$ a subseq of $Kx_n \longrightarrow z \in Z$, $\Rightarrow$ is Cauchy. $\|x_n - x_m\|_X \le c\underbrace{\|Dx_n - Dx_m\|_X}_{=0} + c\underbrace{\|Kx_n - Kx_m\|}_{\text{small}}$ $\Rightarrow$ subseq of x_n is Cauchy.

(2) Pick a closed subspace $V \subseteq X$ s.t. $\ker D \oplus V = X$. im $D = D(V)$.

Spse $x_n \in V$ s.t. $Dx_n \longrightarrow y \in Y$. claim: x_n is bdd.

If not, $D\left(x_n/\|x_n\|_X\right) = \frac{1}{\|x_n\|_X} Dx_n \longrightarrow 0$, $K\left(x_n/\|x_n\|_X\right)$ has a conv. subseq. $\overset{\text{(as in (1))}}{\Longrightarrow}$ $x_n/\|x_n\|_X$ has a subseq. $\longrightarrow x \in V$, $\|x\|_X = 1$, but $Dx = 0$ contra. since $D|_V$ inj. Now x_n bdd $\Rightarrow$ subseq. Kx_n conv., Dx_n also conv. $\Rightarrow$ x_n has conv. subseq. $\longrightarrow x \in X$, $Dx = y \Rightarrow y \in \text{im } D$.

$\Rightarrow$ im D is closed. ☒

Cor: If $\Gamma = \emptyset$, D is semi-Fredholm.

asymptotic regularity, part 1:

Lemma 2: If $\eta \in L^p(E)$ & $D\eta = \xi \in W^{k-1,p}(F)$, then $\eta \in W^{k,p}(E)$ & $\exists$ $c > 0$ indep of η s.t. $\|\eta\|_{W^{k,p}} \le c\|\xi\|_{W^{k-1,p}} + c\|\eta\|_{L^p}$.

Pf: Already known locally. Suff. to assume $\eta \in W^{k-1,p}(E)$, consider $W^{k,p}$-norm on cyl. ends.

$D\eta - S\eta$ bdd in terms of $\|S\|_{C^{k-1}}$

Lemma 1 $\Rightarrow$ $\|\eta\|_{W^{k,p}((N,N+1)\times S^1)} \le c\|\bar\partial\eta\|_{W^{k-1,p}((N-1,N+2)\times S^1)} + c\|\eta\|_{W^{k-1,p}((N-1,N+2)\times S^1)}$

$c > 0$ is indep. of N since S conv. asymptotically to S_∞.

Sum over N $\Rightarrow$ $\|\eta\|_{W^{k,p}((1,\infty)\times S^1)} \le c\|D\eta\|_{W^{k-1,p}(Z_+)} + c\|\eta\|_{W^{k-1,p}(Z_+)}$. $\square$

bad news: $W^{k,p}(E) \hookrightarrow W^{k-1,p}(E)$ not cpct.

<u>fundamental asymp. result</u>: Assume $A = -J_0 \partial_t - S(t)$ a nondeg asymp. op., let $D := \partial_s - A = \underbrace{\partial_s + J_0 \partial_t}_{\bar\partial} + S(t) : W^{k,p}(\mathbb{R}\times S^1, \mathbb{R}^{2n}) \to W^{k-1,p}(\mathbb{R}\times S^1, \mathbb{R}^{2n})$.

Then D is an iso.

<u>pf</u> (for $p=2$): Lemma 2 $\Rightarrow$ suff. to prove for $k=1$.

Complexify & consider $\partial_s + J_0 \partial_t + S(t) : H^1(\mathbb{R}\times S^1, \mathbb{C}^{2n}) \to L^2(\mathbb{R}\times S^1, \mathbb{C}^{2n})$.

Apply F.T. w.r.t. the $\mathbb{R}$-variable:

$u : \mathbb{R}\times S^1 \to \mathbb{C}^{2n} \rightsquigarrow \mathcal{F}u = \hat u : \mathbb{R}\times S^1 \to \mathbb{C}^{2n}$,

$$u(s,t) = \int_{\mathbb{R}} \hat u(\sigma,t) e^{2\pi i s\sigma} d\sigma, \qquad \hat u(\sigma,t) = \int_{\mathbb{R}} u(s,t) e^{-2\pi i s\sigma} ds.$$

Then $\langle u, v\rangle_{L^2} = \langle \hat u, \hat v\rangle_{L^2}$

$$\widehat{\partial_s u}(\sigma,t) = 2\pi i\sigma\, \hat u(\sigma,t), \qquad \widehat{\partial_t u}(\sigma,t) = \partial_t \hat u(\sigma,t), \qquad \widehat{Su}(\sigma,t) = S(t)\hat u(\sigma,t).$$

Thus $\partial_s u + J_0 \partial_t u + Su = f$ becomes $\underbrace{2\pi i\sigma \hat u + J_0 \partial_t \hat u + S\hat u}_{(2\pi i\sigma - A)\hat u(\sigma,\cdot)} = \hat f$.

crucial observation: $\text{spectrum}(A) \subseteq \mathbb{R}\setminus\{0\} \Rightarrow 2\pi i\sigma - A : H^1(S^1) \to L^2(S^1)$ is invtbl $\forall\, \sigma\in\mathbb{R}$, has inverse $R_\sigma := (2\pi i\sigma - A)^{-1}$.

$\rightsquigarrow$ given $\hat f \in L^2$, $\exists$ candidate sol. to $Du = f$ given by

$\hat u(\sigma,\cdot) = R_\sigma(\hat f(\sigma,\cdot))$. (for almost all $\sigma\in\mathbb{R}$).

<u>to check</u>: $f \mapsto u$ defines a bdd lin. map $L^2(\mathbb{R}\times S^1) \to H^1(\mathbb{R}\times S^1)$.

<u>note</u>: $\|u\|_{H^1}$ is equiv. to $\|\hat u\|_{L^2} + \||\sigma|\cdot\hat u\|_{L^2} + \|\partial_t \hat u\|_{L^2}$ ☐