<u>Recall</u>: $D_r = D + rB$ for $B\eta = \beta\bar{\eta}$ $\left(\beta \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(\bar{E}, F))\right)$ satisfies

(W): $D_r^* D_r \eta = D^* D\eta + r^2 B^* B\eta + rB_1\eta$ for some $B_1 : E \to E$.

Can always choose β & area form dvol on $\dot{\Sigma}$ & perturb D on a cpct subset s.t.

(i) $|\beta|$ & $\frac{1}{|\beta|}$ are bdd outside a cpct subset

(ii) $\beta^{-1}(0)$ finite & near each $\zeta \in Z(\beta) = Z_+(\beta) \amalg Z_-(\beta) := \beta^{-1}(0)$,

$\exists$ a nbhd $\mathcal{D}(\zeta) \cong \mathbb{D}$ w/ coords & trivs. s.t. $j = i$, dvol $= ds \wedge dt$,

$$\beta(z) = \begin{cases} z & \text{if } \zeta \in Z_+(\beta) \\ \bar{z} & \text{if } \zeta \in Z_-(\beta). \end{cases}$$

(iii) On the nbhds $\mathcal{D}(\zeta)$, $D = \bar{\partial}$ in the coords/trivs.

$\Rightarrow$ On $\mathcal{D}(\zeta)$, $D_r \eta = 0 \iff \begin{cases} \bar{\partial}\eta + rz\bar{\eta} = 0 & \text{if } \zeta \in Z_+(\beta) \\ \bar{\partial}\eta + r\bar{z}\bar{\eta} = 0 & \text{if } \zeta \in Z_-(\beta). \end{cases}$

concentration lemma: Spse $r_n \to \infty$, $\eta_n \in \ker D_{r_n}$ w/ $\|\eta_n\|_{L^2}$ bdd,

consider for each $\zeta \in Z_\pm(\beta)$, $f_n^\zeta : \mathbb{D}_{\sqrt{r_n}} \to \mathbb{C} : z \mapsto \frac{1}{\sqrt{r_n}} \eta\left(\frac{z}{\sqrt{r_n}}\right)$

in the coords/trivs. on $\mathcal{D}(\zeta)$. Then:

(1) $\|f_n^\zeta\|_{L^2(\mathbb{D}_{\sqrt{r_n}})} = \|\eta_n\|_{L^2(\mathcal{D}(\zeta))}$.

(2) $\begin{cases} \bar\partial f_n^\zeta + z \bar{f}_n^\zeta = 0 & \text{if } \zeta \in Z_+(\beta) \\ \bar\partial f_n^\zeta + \bar{z} \bar{f}_n^\zeta = 0 & \text{if } \zeta \in Z_-(\beta) \end{cases}$

(3) f_n^ζ has a C^∞_{loc}-conv. subseq. $f_n^\zeta \to f_\infty^\zeta \in C^\infty(\mathbb{C}) \cap L^2(\mathbb{C})$.

(4) For any other seq. $\xi_n \in \ker D_{r_n}$ under some assumptions & resulting rescaled seqs. g_n^ζ, $\lim_{n\to\infty} \langle \eta_n, \xi_n \rangle_{L^2} = \sum_{\zeta \in Z(\beta)} \langle f_\infty^\zeta, g_\infty^\zeta \rangle_{L^2(\mathbb{C})}$.

Pf: (1) + (2) computations $\Longrightarrow$ (via L^2-bound + elliptic reg.) (3).

(4) follows from $\langle \eta_n, \xi_n \rangle_{L^2(\mathcal{D}(\zeta))} = \langle f_n^\zeta, g_n^\zeta \rangle_{L^2(\mathbb{D}_{\sqrt{r_n}})}$ after

claim: For $\dot\Sigma_\varepsilon := \dot\Sigma \setminus \bigcup_{\zeta \in Z(\beta)} \mathcal{D}(\zeta)$, $\|\eta_n\|_{L^2(\dot\Sigma_\varepsilon)} \to 0$,

i.e. all "energy" of η_n is concentrated near $Z(\beta)$.

lemma: On $\dot\Sigma_\varepsilon$, $|B\eta| \geq c|\eta|$.

$$0 = \|D_{r_n}\eta_n\|^2_{L^2} = \langle \eta_n, D_{r_n}^* D_{r_n} \eta_n \rangle_{L^2}$$

$$= \langle \eta_n, D^* D \eta_n \rangle_{L^2} + r_n^2 \langle \eta_n, B^* B \eta_n \rangle_{L^2} + r_n \langle \eta_n, B_1 \eta_n \rangle_{L^2}$$

$$\geq r_n^2 \langle B\eta_n, B\eta_n \rangle_{L^2(\dot\Sigma_\varepsilon)} - r_n |\langle \eta_n, B\eta_n \rangle_{L^2}|$$

$$\geq r_n^2 c^2 \|\eta_n\|^2_{L^2(\dot\Sigma_\varepsilon)} - r_n c_1 \|\eta_n\|_{L^2(\dot\Sigma)}$$

$$\Rightarrow \|\eta_n\|^2_{L^2(\dot\Sigma_\varepsilon)} \leq \frac{c_1}{c^2 r_n} \|\eta_n\|_{L^2(\dot\Sigma)} \to 0 \quad \text{as } n \to \infty. \qquad \square$$

Let $f := f^3_\infty \in C^\infty(\mathbb{C}) \cap L^2(\mathbb{C})$ as in lemma; this satisfies

$$D_+ f := \bar\partial f + z\bar f = 0 \quad \text{or} \quad D_- f := \bar\partial f + \bar z \bar f = 0$$

$\Rightarrow$ $\exists$ Weitzenböck formulas relating $D_\pm^* D_\pm$ to $\bar\partial^* \bar\partial = -\partial\bar\partial = -\partial_s^2 - \partial_t^2$
$= -\Delta$.

Recall: $u : \mathbb{C} \to \mathbb{R}$ is subharmonic if $-\Delta u \le 0$.

$\Rightarrow$ satisfies mean value property $u(z_0) \le \frac{1}{\pi r^2} \int_{\mathbb{D}_r(z_0)} u \quad \forall\, r > 0,\ z_0 \in \mathbb{C}$.

EX: For $f \in C^\infty(\mathbb{C}, \mathbb{C})$, $\Delta |f|^2 = 2\,\mathrm{Re}\underbrace{\langle f, \Delta f\rangle}_{\text{Hermitian inner prod. } \mathbb{C}} + 2|\nabla f|^2$.

prop 1: All sols. $f \in L^2(\mathbb{C})$ to $D_- f = 0$ are trivial.

Pf: Weitzenböck: $D_-^* D_- f = -\Delta f + |z|^2 f \quad \forall\, f \in C^\infty(\mathbb{C}, \mathbb{C})$.

$D_- f = 0 \Rightarrow -\Delta f = -|z|^2 f \Rightarrow -\Delta |f|^2 = -2\,\mathrm{Re}\langle f, |z|^2 f\rangle - 2|\nabla f|^2 \le 0$

$\Rightarrow |f|^2$ is subharmonic. $\Rightarrow \forall\, z_0 \in \mathbb{C},\ r > 0$,

$\pi r^2 |f(z_0)|^2 \le \int_{\mathbb{D}_r(z_0)} |f|^2 \le \|f\|^2_{L^2} \Rightarrow |f(z_0)|^2 = 0$. ☒

prop 2: All sols. $f \in L^2(\mathbb{C})$ to $D_+ f = 0$ are real multiples of $e^{-\frac{1}{2}|z|^2}$.

pf sketch: By a similar Weitzenböck + MVP argument, $\mathrm{Im}\, f \equiv 0$.

Then $f(z) = g(z) e^{-\frac{1}{2}|z|^2}$ for some $g : \mathbb{C} \to \mathbb{R}$

Leibniz: $0 = D_+ f = D_+\left(g e^{-\frac{1}{2}|z|^2}\right) = (\bar\partial g) e^{-\frac{1}{2}|z|^2} + g \underbrace{D_+\left(e^{-\frac{1}{2}|z|^2}\right)}_{=0}$

$\Rightarrow \bar\partial g \equiv 0 \Rightarrow g \equiv \text{const}$. ☒

Concentration lemma $\Rightarrow$ L^2 - bdd seq. $\eta_n \in \ker D_{r_n}$ has subseq.

"converging" to an element of $\bigoplus_{\zeta \in Z_+(\beta)} \underbrace{\ker D_+}_{\text{1-dim. space spanned by } e^{-\frac{1}{2}|z|^2}}$,

& L^2-products are preserved in the limit.

$\Rightarrow$ Cor: For $r \gg 0$, $\dim \ker D_r \le \dim \bigoplus_{\zeta \in Z_+(\beta)} \ker D_+ = \# Z_+(\beta)$.

Same arg. for formal adjoint $\Rightarrow$ $\dim \ker D_r^* \le \# Z_-(\beta)$ for $r \gg 0$.

Still to prove: If either $Z_+(\beta)$ or $Z_-(\beta)$ is empty, these are equalities for $r \gg 0$.

<u>linear gluing argument</u>: Fix $\rho \in C_0^\infty(\mathring{\mathbb{D}}, [0,1])$, $\rho|_{\mathbb{D}_{1/2}} \equiv 1$.

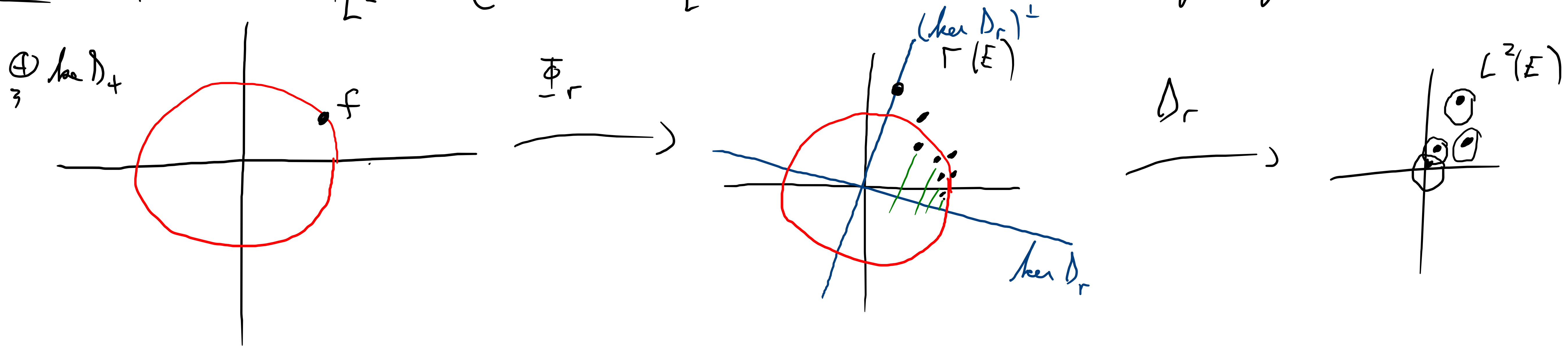
$\zeta \in Z_+(\beta) \rightsquigarrow \Phi_r^\zeta : \ker D_+ \longrightarrow \Gamma(E) : c e^{-\frac{1}{2}|z|^2} \longmapsto \rho(z)\, c \sqrt{r}\, e^{-\frac{1}{2} r |z|^2}$

in coords. on $\mathcal{D}(\zeta)$, 0 everywhere else.

$\rightsquigarrow \Phi_r : \bigoplus_{\zeta \in Z_+(\beta)} \ker D_+ \longrightarrow \Gamma(E)$ nearly an L^2-isometry s.t. $D_r \Phi_r(f) \approx 0$ for $r >> 0$.

"<u>pregluing map</u>"

<u>EX</u>: $\| D_r \Phi_r f \|_{L^2} \leq e^{-cr} \|f\|_{L^2}$ for some $c > 0$ indep. of r.



<u>idea</u>: If D_r is "suff. injective" on $(\ker D_r)^\perp \Rightarrow$ can find ! <u>small</u>

$\xi \in (\ker D_r)^\perp$ s.t. $D_r(\Phi_r f + \xi) = 0$ for $r >> 0$

& $\Phi_r f + \xi \neq 0$. $\quad L^2$-ortho proj. to $\ker D_r$

<u>Lemma</u>: If $Z_-(\beta) = \emptyset$, $\forall r >> 0$, $\Pi_r \Phi_r : \bigoplus_{\zeta \in Z_+(\beta)} \ker D_+ \longrightarrow \ker D_r$ is <u>injective</u>.

<u>Pf</u>: Uses Weitzenböck to get a unif. bound on $\| (D_r|_{(\ker D_r)^\perp})^{-1} \|$

indep. of r. $\boxtimes$

pf of index thm: D_r is asymp. at $z \in \Gamma_\pm$ to

$A_{z,r}\eta := A_z \eta - r\beta_z \bar\eta$ for some $\beta_z : \bar E_z \to E_z$ nowhere zero.

In an asymp. triv. τ, $A_{z,r}\eta = -i\partial_t \eta - S_z(t)\eta - r\beta_z^\tau(t)\bar\eta$

for some $\beta_z^\tau : S^1 \to \mathbb{C}\setminus\{0\}$.

EX: $\# Z(\beta) = c_1^\tau(\mathrm{Hom}_{\mathbb{C}}(\bar E, F)) + \sum_{z\in\Gamma^+} \mathrm{wind}(\beta_z^\tau) - \sum_{z\in\Gamma^-} \mathrm{wind}(\beta_z^\tau)$.

Assume: Can choose β s.t. $A_{z,r}$ nondeg. $\forall z \in \Gamma$ $\forall r \geq 0$.

$\Rightarrow D_r$ Fredholm $\forall r \geq 0 \Rightarrow \mathrm{ind}(D) = \mathrm{ind}(D_r) \overset{(r\gg 1)}{=} \# Z(\beta)$

$= c_1^\tau(E \otimes T\dot\Sigma \otimes E) + \text{winding terms} = \chi(\dot\Sigma) + 2c_1^\tau(E) + \text{winding terms}$.

final lemma: $\forall k \in \mathbb{Z}$, $\exists$ asymp. op. $A_k = -i\partial_t - S_k(t)$ on the trivial line bundl & $\beta_k : S^1 \to \mathbb{C}\setminus\{0\}$ s.t.

$A_{k,r}\eta := A_k\eta - r\beta\bar\eta$ is nondeg. $\forall r \geq 0$ &

$\mu_{CZ}(A_{k,r}) = \mathrm{wind}(\beta_k) = k$.

pf for $k=0$: $-i\partial_t - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ works for $k=0$. ∎