geometric setting for hol. curves in SFT

Recall: $M^{2n-1} \subseteq (W^{2n}, \omega) \Rightarrow \omega_M := \omega|_{TM} \in \Omega^2(M)$ has maximal rank

($\Leftrightarrow \exists$ 1-dim subbndl $\ell_\omega := \ker \omega \subseteq TM$)

defn: M an oriented $(2n-1)$-mfd:

- **Hamiltonian structure** on M := closed 2-form ω of maximal rank

  ($\rightsquigarrow$ characteristic line field $\ell_\omega = \ker \omega \subseteq TM$.

  ω descends to a nondeg. 2-form on $TM/\ell_\omega \Rightarrow \ell_\omega$ has a canonical orientation.)

- **framing** of ω: $\lambda \in \Omega^1(M)$ s.t. $\lambda|_{\ell_\omega} > 0$ $\quad (\Leftrightarrow \lambda \wedge \omega^{n-1} > 0)$

$\rightsquigarrow$ - **Reeb vector fld**: $R \in \Gamma(\ell_\omega)$ s.t. $\lambda(R) \equiv 1$.

- co-oriented symplectic hyperplane bndl $\xi := \ker \lambda$ with symp. str. $\omega|_\xi$.

$$\mathcal{L}_R \omega = d \underbrace{\iota_R \omega}_{=0} + \iota_R \underbrace{d\omega}_{=0} = 0 \Rightarrow$$

prop: $\exists!$ sympl. connection ∇^ω on $(\xi, \omega|_\xi)$ along each integral curve of ℓ_ω

s.t. parallel transp. = linearized Reeb flow composed w. the proj

$$\pi_\xi : TM = \mathbb{R}R \oplus \xi \longrightarrow \xi. \qquad \boxtimes$$

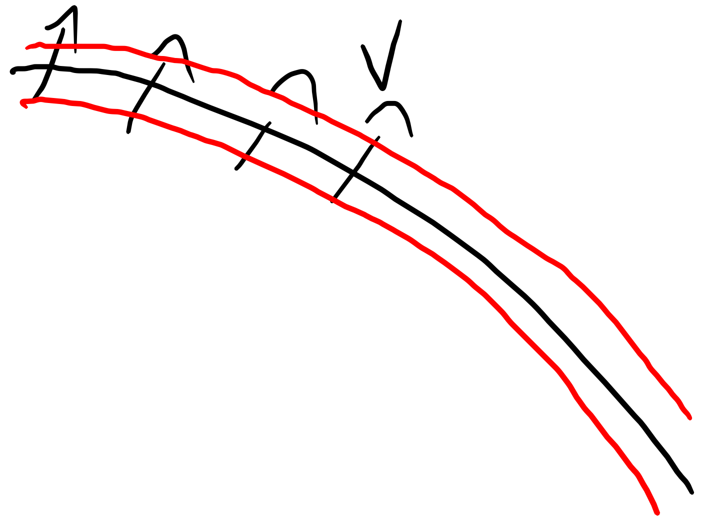
defn: A periodic orbit of R parametrized by $\gamma : S^1 \to M$ is **nondegenerate**

if $\nexists$ nontrivial $\eta \in \Gamma(\gamma^*\xi)$ s.t. $\nabla^\omega_t \eta \equiv 0$.

$\Leftrightarrow \ker A_\gamma = \{0\}$ for $A_\gamma := -J\nabla^\omega_t : \Gamma(\gamma^*\xi) \circlearrowleft$ $\quad$ ($J : \xi \to \xi$ cpx str. compat. w. $\omega|_\xi$).

prop: Any vec. fld V pos. $\pitchfork$ to the oriented hypersurface $M^{2n-1} \subseteq (W^{2n}, \omega)$ induces a framing $\lambda := \omega(V, \cdot)|_{TM}$ of $\omega_M := \omega|_{TM}$, & $\exists$ tubular nbhd $M \subseteq (\mathcal{N}(M), \omega) \cong (\underset{r}{(-\varepsilon,\varepsilon)} \times M, \ \omega_M + d(r\lambda))$ s.t. $M = \{0\} \times M$.

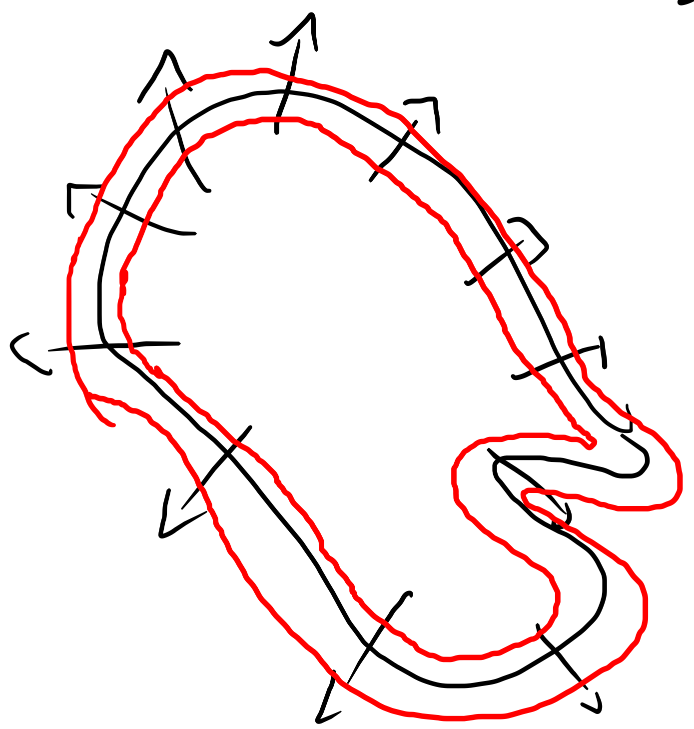
pf: $\Phi : (-\varepsilon,\varepsilon) \times M \hookrightarrow W : (r,x) \longmapsto \varphi_V^r(x)$, $\Phi^*\omega$ satisfies



$$\Phi^*\omega|_{TM} = \omega_M, \quad \Phi^*\omega(\partial_r, \cdot)|_{TM} = \omega(V,\cdot)|_{TM} = \lambda$$

$\Rightarrow$ along $M = \{0\} \times M$, ω matches $\omega_M + dr \wedge \lambda$
$= \omega_M + d(r\lambda)$ (since $r = 0$).

$\Rightarrow \forall\, t \in [0,1]$, $t\Phi^*\omega + (1-t)(\omega_M + d(r\lambda))$ is symp. near M, Moser it! $\boxtimes$



defn: closed $M^{2n-1} \subseteq (W^{2n}, \omega)$ is *stable* if $\exists$ near M a "*stabilizing vec. fld*" $V \pitchfork M$ s.t. $\forall\, t$ near 0,
$$M \xrightarrow[\cong]{\varphi_V^t} M_t := \varphi_V^t(M)$$ preserves char. line field.

ex: $M \subseteq (W,\omega)$ of ctct type, Liouville vec. fld is stabilizing.

__defn__: (ω,λ) is a __stable Ham. str.__ (SHS) on M^{2n-1} if ω is a Ham. str. & λ is a framing s.t. $d\lambda(R,\cdot)\equiv 0$ (i.e. $\ker\omega\subseteq\ker d\lambda$)

λ is a "stable framing".

__prop__: $M^{2n-1}\subseteq(W^{2n},\omega)$ is stable $\Longleftrightarrow$ ω_M admits a stable framing.

__pf__: $\underline{\Rightarrow}$: V stab. vec. fld $\Rightarrow$ $(\varphi_V^t)^*\omega|_{TM}$ has same kernel $\forall t$

$\Rightarrow$ $\mathcal{L}_V\omega(R,\cdot)|_{TM}\equiv 0=(d\iota_V\omega+\iota_V d\omega)(R,\cdot)|_{TM}=d\lambda(R,\cdot)=0$

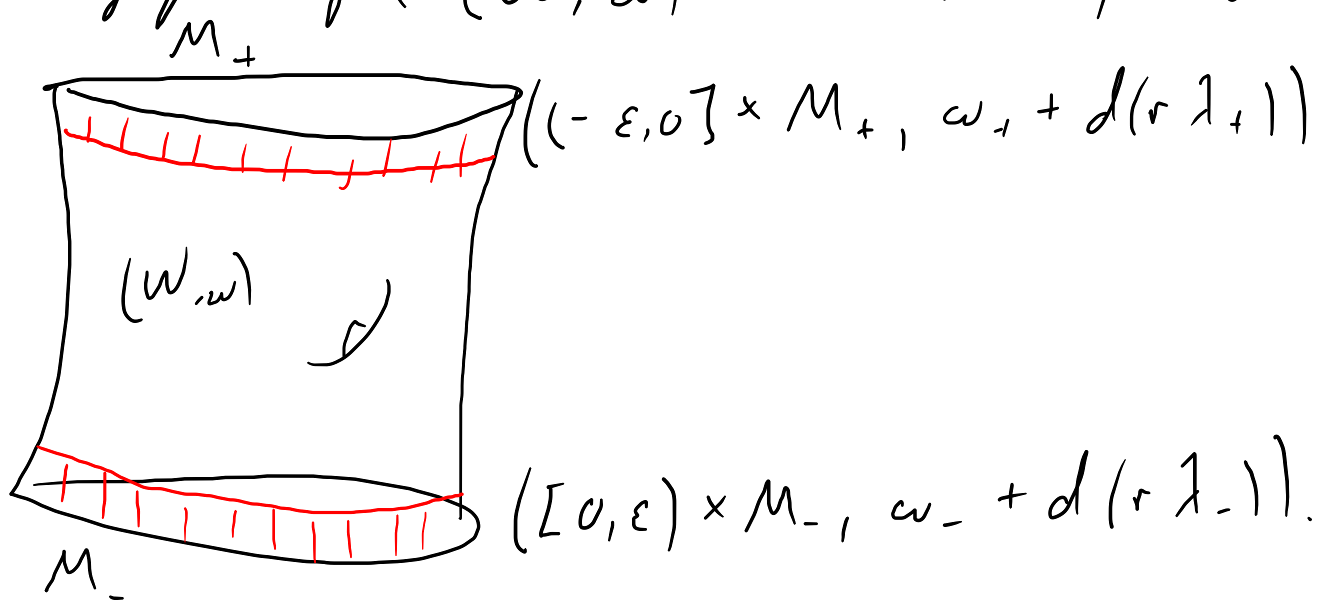
for $\lambda:=\iota_V\omega|_{TM}$.

$\underline{\Leftarrow}$: nbhd $\cong((-\varepsilon,\varepsilon)\times M,\ \omega_M+d(r\lambda))$, then $M_t:=\{t\}\times M$ has

$(\omega_M+d(r\lambda))|_{TM_t}=\omega_M+t\,d\lambda$ also kills R if $d\lambda(R,\cdot)\equiv 0$. ☒

__defn__: $M^{2n-1}_\pm$ closed oriented mflds w/ SHS $\mathcal{H}_\pm:=(\omega_\pm,\lambda_\pm)$.

A __symplectic cobordism__ from $(M_-,\mathcal{H}_-)$ to $(M_+,\mathcal{H}_+)$ is a cpct symp. mfld (W,ω) with bndry $\partial W\cong -M_-\amalg M_+$ s.t. $\omega|_{TM_\pm}=\omega_\pm$.



__EX__: (ω,λ) SHS $\Rightarrow$ (a) linearized flow of R preserves $\xi=\ker\lambda$

(b) $\gamma:S^1\to M$ T-periodic orbit $\Rightarrow$ $A_\gamma\eta=-J(\nabla_t\eta-T\nabla_\eta R)$

for any symmetric conn. ∇ on M.

<u>symplectization</u>: M^{2n-1} w/ framed Ham. str. $\mathcal{H} = (\omega, \lambda)$

$\Rightarrow \big((-\varepsilon, \varepsilon) \times M, \ \omega + d(r\lambda) \big)$ is symp. for $\varepsilon > 0$ suff. small

$\Rightarrow (\mathbb{R} \times M, \ \omega_\varphi)$ is symp. for $\omega_\varphi := \omega + d(\varphi(r)\lambda)$,

$\varphi \in \mathcal{T} := \{ \varphi \in C^\infty(\mathbb{R}, (-\varepsilon, \varepsilon)) \mid \varphi' > 0 \}$.

<u>defn</u>: $\mathcal{J}(\mathcal{H}) := \{ J : T(\mathbb{R} \times M) \circlearrowleft \mid J^2 = -\mathrm{Id}$ s.t.

(1) inv't. under translation $(r, x) \longmapsto (r + c, x) \ \forall c \in \mathbb{R}$,

(2) $T(\mathbb{R} \times M) \overset{\mathbb{C}\text{px}}{=} \epsilon \oplus \xi$ where $\epsilon := \mathrm{Span}\{\partial_r, R\}$

$J(\partial_r) = R, \ J(R) = -\partial_r$

& $J|_\xi$ compatible w/ $\omega|_\xi$.

<u>EX</u>: $J \in \mathcal{J}(\mathcal{H})$ is tamed by $\omega_\varphi \ \forall \varphi \in \mathcal{T} \iff \lambda$ is stable.

<u>defn</u>: For $\mathcal{H} = (\omega, \lambda)$ is an SAS & $J \in \mathcal{J}(\mathcal{H})$, the <u>energy</u> of a J-hol. curve $u: (\Sigma, j) \longrightarrow (\mathbb{R} \times M, \sigma)$ is $E(u) := \sup_{\varphi \in \hat{\mathcal{T}}} \int_\Sigma u^* \omega_\varphi$.

Then $E(u) \geq 0$, $=$ iff $u = \text{const}$.

<u>ex 1</u>: $x: \mathbb{R} \longrightarrow M$ a T-periodic orbit of R $\rightsquigarrow$ $\gamma: S^1 \longrightarrow M: t \longmapsto x(Tt)$,

$\rightsquigarrow$ <u>trivial cylinder</u> over γ: $u_\gamma: \mathbb{R} \times S^1 \longrightarrow \mathbb{R} \times M: (s,t) \longmapsto (Ts, \gamma(t))$.

$$\partial_s u_\gamma + J \partial_t u_\gamma = T \cdot \partial_r + J(T \cdot R(\gamma)) = 0 \implies u_\gamma \text{ is } J\text{-hol.},$$

$$E(u_\gamma) = \sup_{\varphi \in \hat{\mathcal{T}}} \int_{\mathbb{R} \times S^1} [\underbrace{u_\gamma^* \omega}_{=0} + u_\gamma^* d(\varphi(r)\lambda)] \overset{(\text{Stokes})}{=} 2\varepsilon T.$$

<u>EX</u>: $x: \mathbb{R} \longrightarrow M$ any orbit of R, $u: \mathbb{C} \longrightarrow \mathbb{R} \times M: s + it \longmapsto (s, x(t))$ is also J-hol., but $E(u) = \infty$.

prop (computation): The linearized CR-op. for $u_\gamma : \mathbb{R}\times S^1 \to \mathbb{R}\times M$ is given w.r.t. splitting $u_\gamma^* T(\mathbb{R}\times M) = u_\gamma^* \epsilon \oplus u_\gamma^* \xi$ by

$$(D_{u_\gamma}\eta)\partial_s = \left(\partial_s - \begin{pmatrix} -i\partial_t & T\cdot d\lambda(R(\gamma),\cdot) \\ 0 & A_\gamma \end{pmatrix}\right)\eta .$$

$\Rightarrow$ If λ is stable, this is $\partial_s - \underbrace{(-i\partial_t \oplus A_\gamma)}_{\text{asymp. op. on } \gamma^*\epsilon \oplus \gamma^*\xi}$. ⊠

examples:

(1) "ctct case": α = ctct form on $M \Rightarrow (d\alpha, \alpha)$ is an SHS
$R = R_\alpha$ usual Reeb fld from ctct geom., $\mathcal{J}(\alpha) := \mathcal{J}(\mathcal{H})$.

(2) "Floer case": (W^{2n}, Ω) closed symp mfd, $H : S^1 \times W \to \mathbb{R}$, $H_t := H(t,\cdot) : W \to \mathbb{R}$
let $M := S^1 \times W$, $\mathcal{H} := (\omega, \lambda) := (\Omega + dt\wedge dH, dt)$
Think of M as trivial fibration $W \hookrightarrow M \to S^1$, then $\omega|_{\text{fibers}}$ symp.
$\Rightarrow \omega$ is a Ham. str., $\underset{``\lambda\wedge\omega^n"}{dt\wedge\Omega^n} > 0 \Rightarrow \lambda$ is a framing,
$d\lambda = 0 \Rightarrow \lambda$ is stable.
$\lambda(R) \equiv 1 \Rightarrow R = \partial_t + X_t$ for some t-dep. vec. fld X_t on W,
$\omega(R,\cdot)|_{\text{fibers}} = 0 = \Omega(X_t,\cdot) + dt\wedge dH(\partial_t,\cdot) = \Omega(X_t,\cdot) + dH_t$
$\Rightarrow X_t$ = Ham. vec. fld of H_t. $\left\{\begin{matrix}\text{Reeb orbits homologous} \\ \text{to } [S^1\times\{\text{const}\}] \in H_1(M)\end{matrix}\right\} = \left\{\begin{matrix}\text{1-per. orbits} \\ \text{of } X_{H_t}\end{matrix}\right\}$.

Any $J \in \mathcal{J}(\mathcal{H})$ is equivalent to a choice of family $\{J_t \in \mathcal{J}(W,\Omega)\}_{t\in S^1}$.

EX: If $u = (\varphi, v) : \mathbb{R}\times S^1 \to \mathbb{R}\times M = (\mathbb{R}\times S^1)\times W$ is J-hol., then $\varphi : \mathbb{R}\times S^1 \to \mathbb{R}\times S^1$ is hol. In case $\varphi = \mathrm{Id}$,
$v : \mathbb{R}\times S^1 \to W$ then sats. Floer eqn: $\partial_s v + J_t(v)(\partial_t v - X_t(v)) = 0$.