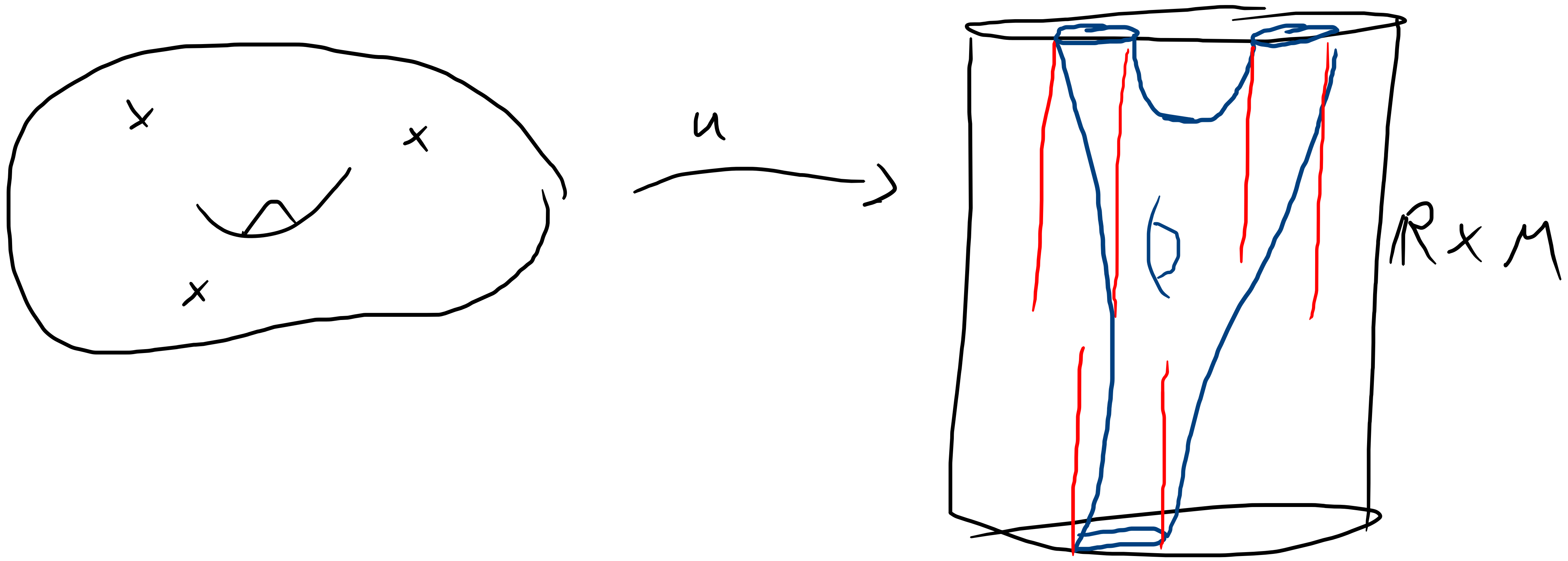
- Lecture 6 in full by end of this week
- videos will not stay up forever

$(\dot{\Sigma} := \Sigma \setminus (\Gamma^+ \cup \Gamma^-), j)$ punctured R.S., $z \in \Gamma^\pm \rightsquigarrow$ cylindrical end $\dot{U}_z \cong Z_\pm$

M^{2n-1} closed mfd w/ SHS $\mathcal{H} = (\omega, \lambda)$, $J \in \mathcal{J}(\mathcal{H})$. $\quad (Z_+ = [0,\infty) \times S^1,\ Z_- = (-\infty, 0] \times S^1)$



Consider J-hol. curves

$u : (\dot{\Sigma}, j) \longrightarrow (\mathbb{R} \times M, J)$

asymptotically cylindrical:

$\forall\ z \in \Gamma^\pm$, $\exists$ Reeb orbit

$\gamma_z : S^1 \longrightarrow M$

($\rightsquigarrow$ J-hol. trivial cylinder $u_z : \mathbb{R} \times S^1 \longrightarrow \mathbb{R} \times M$) s.t. in hol. cyl. coords. $(s,t) \in Z_\pm \cong \dot{U}_z$ for $|s| \gg 0$,

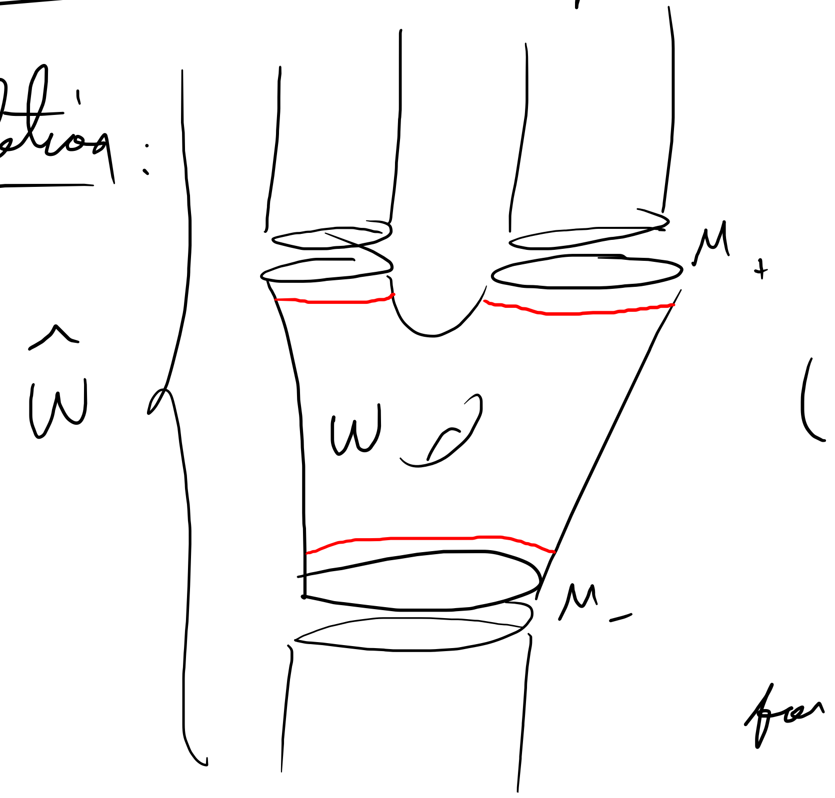
$$\boxed{u(s - s_0, t - t_0) = \exp_{u_z(s,t)} h_z(s,t)}$$

for some const. $s_0 \in \mathbb{R}$, $t_0 \in S^1$, $\quad h_z \in \Gamma(u_z^* T(\mathbb{R} \times M))$ s.t.

$h_z(s, \cdot) \xrightarrow{C^\infty(S^1)} 0$ as $s \to \pm\infty$. $\quad$ ($h_z =:$ "asymptotic representative of u at z")

cobordism: (W,ω) symp. cobordism w/ stable bndy $\partial W = -M_- \sqcup M_+$ w/ SHSs $\mathcal{H}_\pm = (\omega_\pm, \lambda_\pm)$

completion:



$$(\widehat{W}, \omega_\varphi) := ((-\infty,0]\times M_-, \omega_- + d(\varphi(r)\lambda_-)) \cup_{M_-} (W,\omega) \cup_{M_+} ([0,\infty)\times M_+, \omega_+ + d(\varphi(r)\lambda_+))$$

for $\varphi \in \mathcal{T} := \{\varphi \in C^\infty(\mathbb{R}, \underbrace{(-\varepsilon,\varepsilon)}_{\text{small}}) \mid \varphi' > 0$ & $\varphi(r) = r$ near $r=0\}$.

$\mathcal{J}_\tau(\omega, \mathcal{H}_+, \mathcal{H}_-) := \{J : T\widehat{W} \circlearrowleft \mid J^2 = -\mathrm{Id}$, J tamed by ω on W & belongs to $\mathcal{J}(\mathcal{H}_\pm)$ on the cyl. ends$\}$

$u : (\dot\Sigma, j) \to (\widehat{W}, J)$ asymp. cyl. means for $z \in \Gamma$,

$$u(\mathrm{nbhd}(z)) \subseteq \begin{cases} [0,\infty)\times M_+ & \text{if } z\in\Gamma^+ \text{ asymp. to an orbit of } R_+ \text{ in } \{\infty\}\times M_+ \\ (-\infty,0]\times M_- & \text{if } z\in\Gamma^- \quad \text{" } \quad \text{" } R_- \quad \{-\infty\}\times M_- . \end{cases}$$

$E(u) := \sup_{\varphi\in\mathcal{T}} \int_{\dot\Sigma} u^*\omega_\varphi \geq 0$, $=$ iff $u = \text{const}$.

to prove in 3 weeks: $u : (\dot\Sigma, j) \to (\widehat{W}, J)$ is asymp. cyl. $\iff$ $E(u) < \infty$.

relative homology class: Let $\bar\Sigma := \dot\Sigma \cup \left(\bigcup_{z\in\Gamma^\pm} (\{\pm\infty\}\times S^1)\right)$

(compactifying each cylindrical end) $\bar\Sigma$ is a cpt oriented surface w/ bndy

$\overline{W} := \widehat{W} \cup (\{\infty\}\times M_+) \cup (\{-\infty\}\times M_-)$ cpt top. mfd w/ bndy.

$u : \dot\Sigma \to \widehat{W}$ asymp. cyl $\Rightarrow$ extends contin. to $\bar u : \bar\Sigma \to \overline{W}$

$\rightsquigarrow$ $[u] := \bar u_* [\bar\Sigma] \in H_2(\overline{W}, \bar\gamma^+ \cup \bar\gamma^-)$ where $\bar\gamma^\pm := \bigcup_{z\in\Gamma^\pm} \gamma_z(S^1) \subseteq \{\pm\infty\}\times M_\pm \subset \partial\overline{W}$.

$[\bar\Sigma] \in H_2(\bar\Sigma, \partial\bar\Sigma)$

$H_2(\overline{W}, \bar\gamma^+ \cup \bar\gamma^-) = H_2(W, \bar\gamma^+ \cup \bar\gamma^-)$

($\exists$ deformation retraction $\overline{W} \to W$).

__moduli spaces__: Fix $J \in \mathcal{J}(\omega, \mathcal{H}_+, \mathcal{H}_-)$ or $\mathcal{J}(\mathcal{H})$, $g, m \geq 0$,

ordered sets of closed orbits $\gamma^\pm = (\gamma_1^\pm, \dots, \gamma_{k_\pm}^\pm)$ of $R_\pm$,

rel. hom. class $A \in H_2(W, \bar{\gamma}^+ \cup \bar{\gamma}^-)$ for $\bar{\gamma}^\pm := \bigcup_{i=1}^{k_\pm} \gamma_i^\pm(S^1) \subseteq M_\pm \subseteq \partial W$.

$\mathcal{M} := \mathcal{M}_{g,m}(J, A, \gamma^+, \gamma^-) := \{(\Sigma, j, \Gamma^+, \Gamma^-, \Theta, u)\}/\sim$ where:

(1) (Σ, j) = closed R.S of genus g

(2) $\Gamma^\pm = (z_1^\pm, \dots, z_{k_\pm}^\pm)$, $\Theta = (\zeta_1, \dots, \zeta_m)$ disjoint ordered sets of distinct pts. in Σ

(3) $u : (\dot{\Sigma} := \Sigma \setminus (\Gamma^+ \cup \Gamma^-), j) \longrightarrow (\hat{W}, J)$ J-hol., asymp. cylindrical,

s.t. at $z_i^\pm \in \Gamma^\pm$, u is pos/neg asymp. to $\gamma_i^\pm$, $[u] = A$.

(4) $(\Sigma_0, j_0, \Gamma_0^+, \Gamma_0^-, \Theta_0, u_0) \sim (\Sigma_1, j_1, \Gamma_1^+, \Gamma_1^-, \Theta_1, u_1)$ iff

$\exists$ bihol. map

$$\begin{tikzcd} (\Sigma_0, j_0, \Gamma_0^+, \Gamma_0^-, \Theta_0) \arrow[r, "\cong"] \arrow[dr, "u_0"'] & (\Sigma_1, j_1, \Gamma_1^+, \Gamma_1^-, \Theta_1) \arrow[d, "u_1"] \\ & \hat{W} \end{tikzcd}$$

preserving order of ptrs. & Θ ("marked pts.")

$\mathrm{Aut}(u) := \mathrm{Aut}(\Sigma, j, \Gamma_+, \Gamma_-, \Theta, u)$

$:= \{ \varphi : (\Sigma, j) \circlearrowleft \text{ bihol} \mid \varphi|_{\Gamma_+ \cup \Gamma_- \cup \Theta} = \mathrm{Id}$ & $u = u \circ \varphi \}$.

__topology__: $u_n \longrightarrow u \in \mathcal{M}$ means for $n \gg 0$ $\exists$ representatives

$(\Sigma, j_n, \Gamma^+, \Gamma^-, \Theta, u_n)$ s.t. $j_n \xrightarrow{C^\infty(\Sigma)} j$,

fixed

$u_n \xrightarrow{C^\infty_{loc}(\dot{\Sigma})} u$

$\bar{u}_n \xrightarrow{C^0(\bar{\Sigma})} \bar{u}$

__prop__: $ev : \mathcal{M} \longrightarrow \hat{W}^{\times m} : [(\Sigma, j, \Gamma^+, \Gamma^-, \Theta, u)] \longmapsto (u(\zeta_1), \dots, u(\zeta_m))$

is contin. $\boxtimes$

asymptotic regularity: Fix $\delta > 0$ small, let $W^{k,p,\delta}(Z_\pm) := \{ e^{\mp \delta s} f \mid f \in W^{k,p}(Z_\pm) \}$

$$\|f\|_{W^{k,p,\delta}} := \|e^{\pm \delta s} f\|_{W^{k,p}} .$$

lemma: $\forall\, k \in \mathbb{N}$, $1 < p < \infty$, $\delta > 0$ suff. small,

(1) $u \in \mathcal{M} \Rightarrow \forall\, z \in \Gamma^\pm$, if γ_z is nondeg., then the asymp. repr.

$$h_z \in W^{k,p,\delta}(Z_\pm).$$

(2) $u_n \longrightarrow u \in \mathcal{M} \Rightarrow \forall\, z \in \Gamma^\pm$, asymp. reprs $(h_n)_z \xrightarrow{W^{k,p,\delta}(Z_\pm)} h_z$.

pf sketch: Write $u(s,t) = \exp_{u_\gamma(s,t)} h(s,t)$ for $(s,t) \in Z_+$, γ nondeg.

Can reparametrize u s.t. $h(s,t) \in \xi_{\gamma(t)}$, then show:

h satisfies a linear CR-type eqn. $Dh = 0$ where the op. D is C^∞-asymp. to $A_\gamma \Rightarrow$ for $s \geq R >> 0$,

$$\|h(s,\cdot)\|_{L^2(S^1)} \leq \|h(R,\cdot)\|_{L^2(S^1)} \cdot e^{-\delta(s-R)}.$$

$\Rightarrow e^{\delta s} h \in L^p(Z_+)$, $(D - \delta) e^{\delta s} h = 0$ where $D - \delta$ is also a CR-op., asymp. to $A_\gamma + \delta \overset{\text{(reg.)}}{\Longrightarrow} e^{\delta s} h \in W^{k,p}(Z_+)$ $\forall\, k \geq 1$. etc.... $\square$

simple vs. multiply covered

thm: $u : (\dot{\Sigma}, j) \to (\hat{W}, J)$ asymp. cyl. & nonconst., then $u = v \circ \varphi$, where:

- $v : (\dot{\Sigma}', j') \to (\hat{W}, J)$ is an asymp. cyl. curve that is embedded outside of fin.-many self-intersections (isolated) & crit. pts. ($dv(z) = 0$).
- $\varphi : (\Sigma, j) \to (\Sigma', j')$ is a hol. map of closed Riem. surfaces w/ $d := \deg(\varphi) \geq 1$.

(case $d = 1$: φ is a diffeo, u is simple, $\Rightarrow$ u is somewhere injective, i.e. $\exists\ z \in \dot{\Sigma}$ s.t. $du(z) \neq 0$ & $u^{-1}(u(z)) = \{z\}$
$\Rightarrow \mathrm{Aut}(u) = \{\mathrm{Id}\}$.)

(case $d > 1$: u is a d-fold branched cover of the simple curve v.)

cor: $|\mathrm{Aut}(u)| \leq d$ if $u = v \circ \varphi$ for v simple & $\deg(\varphi) = d$.

Micallef-White ($\sim$1990)

(1) <u>intersection lemma</u>: $u, v : (\mathbb{D}, i) \longrightarrow (\mathbb{C}^n, J)$ J-hol. w/ $u(0) = v(0)$, then $\exists$ nbhds $\mathcal{U}, \mathcal{V} \subseteq \mathbb{D}$ of 0 s.t. $u(\mathcal{U}) = v(\mathcal{V})$ or

$$u(\mathcal{U} \setminus \{0\}) \cap v(\mathcal{V}) = u(\mathcal{U}) \cap v(\mathcal{V} \setminus \{0\}) = \emptyset.$$

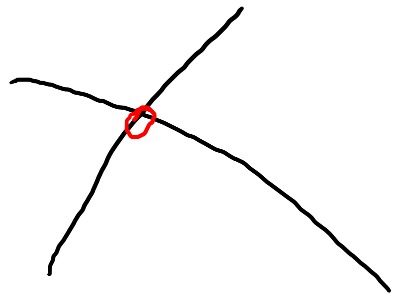
(2) <u>branching lemma</u>: $u : (\mathbb{D}, i) \longrightarrow (\mathbb{C}^n, J)$ J-hol. w/ $du(0) = 0$ but $u \not\equiv \text{const}$, then $\exists$ hol. coord. change near 0, an <u>injective</u> J-hol. map $v : (\mathbb{D}, i) \longrightarrow (\mathbb{C}^n, J)$ & $k \in \mathbb{N}$ s.t. $u(z) = v(z^k)$.

(3) <u>asymptotic lemma</u> (Siefring 2008): (1) & (2) also hold for asymp. cyl. curves $u, v : (\mathbb{D} \setminus \{0\}, i) \longrightarrow (\mathbb{R} \times M, J)$ asymptotic to (covers) of the same orbit.

<u>pf of thm</u>: Let $\mathrm{Crit}(u) = \{ z \in \dot\Sigma \mid du(z) = 0 \}$. (finite)

$\Delta := \{ z \in \dot\Sigma \mid \exists\, z' \in \dot\Sigma$ s.t. $u(z) = u(z')$ but the int. isolated$\}$ (also finite)

$\dot\Sigma' := u(\dot\Sigma \setminus (\mathrm{Crit}(u) \cup \Delta)) \subseteq \hat W$ is a smooth submfd & $J|_{T\dot\Sigma'}$ is a cpx str. on $\dot\Sigma'$ $\rightsquigarrow$ punctured R.S. $(\dot\Sigma', j') \xhookrightarrow{v} (\hat W, J)$



$\exists$ a closed R.S. (Σ', j') s.t. $\dot\Sigma' = \Sigma' \setminus$ fin-many pts, v extends over (Σ', j'), $\varphi : \Sigma \longrightarrow \Sigma'$ def'd by extending $\dot\Sigma \xrightarrow{u} \Sigma'$ over the punctures. $\square$