M closed w/ SHS $\mathcal{H} = (\omega, \lambda) \rightsquigarrow$ Reeb fld R, $\xi = \ker \lambda$

$\mathcal{U} \overset{\text{open}}{\subseteq} M$, $J^{\text{fix}} \in \mathcal{J}(\mathcal{H})$, $\quad\quad \pi_\xi : T(\mathbb{R} \times M) \longrightarrow \xi$ along

$\mathcal{J}_{\mathcal{U}} := \{ J \in \mathcal{J}(\mathcal{H}) \mid J = J^{\text{fix}} \text{ outside } \mathbb{R} \times \mathcal{U} \}$ $\quad\quad \epsilon := \text{Span}\{\partial_r, R\}$

__thm__: $\exists$ comeager $\mathcal{J}_{\mathcal{U}}^{\text{reg}} \subseteq \mathcal{J}_{\mathcal{U}}$ s.t. $\forall J \in \mathcal{J}_{\mathcal{U}}^{\text{reg}}$, all $u \in \mathcal{M}^*(J)$ are Fredholm regular, where $\mathcal{M}^*(J) := \{ u \in \mathcal{M}(J) \mid \exists$ an inj. pt. $z \in \dot{\Sigma}$ s.t. $u(z) \in \mathbb{R} \times \mathcal{U}$, $\wedge$ $\text{im}\, du(z) \cap \xi_{u(z)}^{\perp d\lambda} = \{0\} \}$

where $\xi_{u(z)}^{\perp d\lambda} := \{ X \in T_{u(z)}(\mathbb{R} \times M) \mid d\lambda(X, \cdot)|_{\xi_{u(z)}} = 0 \}$

__ex__: If $\mathcal{H} = (d\alpha, \alpha)$ for a ctct form α in $\mathcal{U}$, condition means

$d\alpha(\pi_\xi \circ du(\cdot), \cdot)|_\xi$, __always__ $\neq 0$ if $\pi_\xi \circ du(z) \neq 0 \in \text{Hom}_{\mathbb{C}}(T\dot{\Sigma}, u^*\xi)$.

__note__: $\pi_\xi \circ du(z) = 0 \iff u$ at z is tangent to ϵ. If true everywhere,

$\Rightarrow u$ is a trivial cylinder (assuming simple).

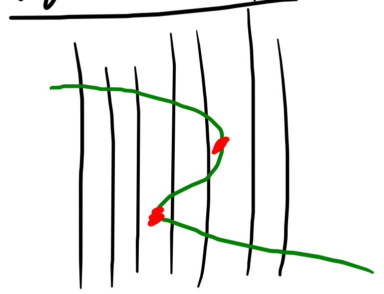
__Lemma 1__: Trivial cylinders over nondeg. Reeb orbits are always regular.

($\Leftarrow$ $\partial_s - A$ on $\mathbb{R} \times S^1$ for a nondeg. asymp. A is invertible).

__note__: $\text{ind}(u_\gamma) = 0$

Lemma 2: If $u : \dot{\Sigma} \longrightarrow \mathbb{R}\times M$ is not (a cover of) a trivial cylinder, then $\pi_\xi \circ du \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(T\dot{\Sigma}, u^*\xi))$ has only isolated zeroes.

pf sketch: $\epsilon = \mathrm{Span}\{\partial_r, R\}$ generates a J-inv't foliation on $\mathbb{R}\times M$.



$\pi_\xi \circ du(z) = 0 \iff$ at z, u is tangent to a leaf of the foliation.

In local coords such a pt., can assume $u = (f, v) : \mathbb{D} \longrightarrow \mathbb{C}\times\mathbb{C}^{n-1}$

$\epsilon = \mathbb{C}\oplus\{0\} \subseteq \mathbb{C}^n$, so u tangent $\epsilon \iff \partial_s v = \partial_t v = 0$.

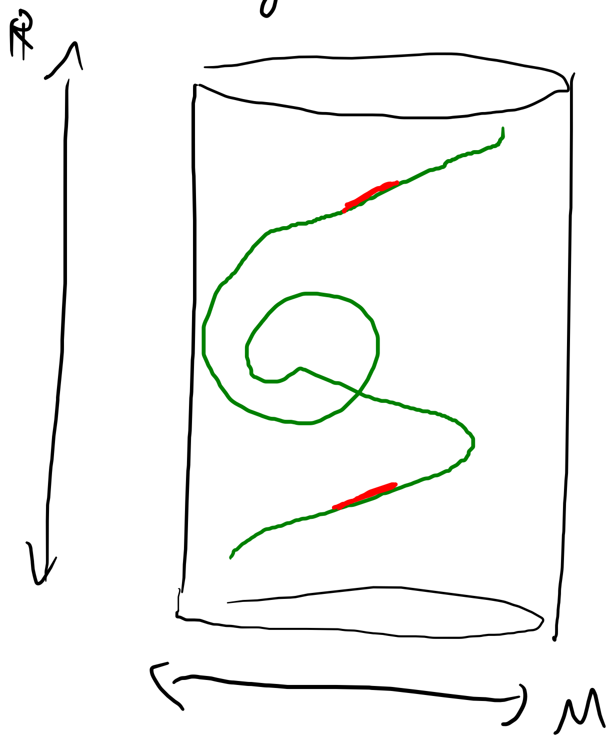
$J = \begin{pmatrix} i & B \\ 0 & J' \end{pmatrix}$ since ϵ is J-inv't $\Longrightarrow \partial_s v + J'(f,v)\,\partial_t v = 0$

$\Longrightarrow \partial_s v$ also sats. a linear CR-type eqn. $\xRightarrow{\text{sim. princ.}}$ zeroes are isolated

unless $\partial_s v \equiv 0$, then $v \equiv$ const $\Rightarrow u$ tangent to a leaf. ∎

Lemma 3: $u = (u_{\mathbb{R}}, u_M) : \dot{\Sigma} \longrightarrow \mathbb{R}\times M$ simple & not a trivial cylinder, then $u_M : \dot{\Sigma} \longrightarrow M$ has a dense set of injective pts.

pf sketch: If inj. pts are not dense, can show $\exists\, z, \zeta \in \dot{\Sigma}$ w/ disjoint nbhds $\mathcal{U}, \mathcal{V} \subseteq \dot{\Sigma}$, s.t. $u_M(\mathcal{U}) = u_M(\mathcal{V})$. ($\Rightarrow u|_{\mathcal{U}} \equiv u|_{\mathcal{V}}$ + $\mathbb{R}$-translation)

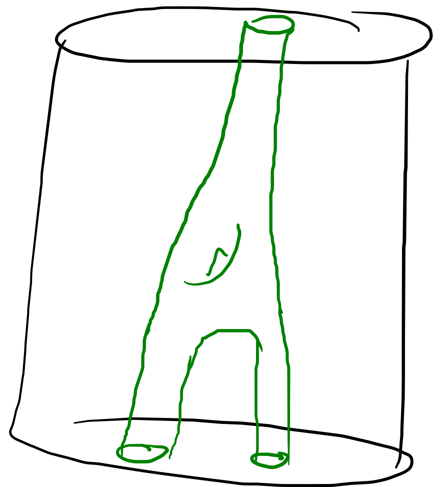


Assuming u is simple, $\Rightarrow \exists\, \tau > 0$ s.t.

$\tau\cdot u := (u_{\mathbb{R}} + \tau, u_M) \equiv u$ up to parametrization.

$\Rightarrow \forall\, k \in \mathbb{Z}$, $k\tau\cdot u \sim u$, as $k \longrightarrow \infty$, in any cpct region of $\mathbb{R}\times M$, image of $k\tau\cdot u$ is arb. close to trivial cylinders

$\Rightarrow u$ is a trivial cyl. ∎



pf of thm:

$\mathcal{J}^\epsilon_\mathcal{U} = \{ C_\epsilon\text{-small perturbations in } \mathcal{J}(\mathcal{H}) \text{ of some chosen } J^{ref} \in \mathcal{J}(\mathcal{H}) \} \subseteq \mathcal{J}_\mathcal{U}$

is a separable Banach mfd

$\rightsquigarrow$ universal moduli space $\mathcal{M}^*(\mathcal{J}^\epsilon_\mathcal{U}) := \{ (u, J) \mid J \in \mathcal{J}^\epsilon_\mathcal{U},\ u \in \mathcal{M}^*(J) \}$

is (up to an action of an aut. group) locally 0-set of $\bar\partial : \mathcal{T} \times \mathcal{B} \times \mathcal{J}^\epsilon_\mathcal{U} \to \mathcal{E}$.

main lemma: $W^{k,p,\delta}(u_0^* T(\mathbb{R}\times M)) \oplus T_{J_0} \mathcal{J}^\epsilon_\mathcal{U} \xrightarrow{L} W^{k-1,p,\delta}(\overline{\mathrm{Hom}}_\mathbb{C}(T\dot\Sigma, u_0^* T(\mathbb{R}\times M)))$

$$(\eta, Y) \longmapsto D_{u_0}\eta + Y(u_0) \circ du_0 \circ j_0$$

is surjective $\forall\ (j_0, u_0, J_0) \in \bar\partial^{-1}(0)$.

pf for $k=1$: If not surj., $\exists\ \theta \neq 0 \in L^{q,-\delta}$ $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$ s.t.

$$\begin{cases} \langle D_{u_0}\eta, \theta \rangle_{L^2} = 0 & \forall\ \eta \in W^{1,p,\delta} \implies \theta \in C^\infty \text{ \& has isolated zeroes.} \\ \langle Y(u_0) \circ du_0 \circ j_0, \theta \rangle_{L^2} = 0 & \forall\ Y \in T_{J_0} \mathcal{J}^\epsilon_\mathcal{U}. \end{cases}$$

trouble: Y is more constrained: $u_0^* T(\mathbb{R}\times M) = u_0^* \epsilon \oplus u_0^* \xi$, $\quad D_{u_0} = \begin{pmatrix} D^\epsilon_{u_0} & D^{\epsilon\xi}_{u_0} \\ D^{\xi\epsilon}_{u_0} & D^\xi_{u_0} \end{pmatrix}$

$Y = \begin{pmatrix} 0 & 0 \\ 0 & Y^\xi \end{pmatrix}$, $\quad \theta = \begin{pmatrix} \theta^\epsilon \\ \theta^\xi \end{pmatrix} \implies \langle Y^\xi(u_0) \circ du_0 \circ j_0, \theta^\xi \rangle_{L^2} = 0 \quad \forall\ Y \in T_{J_0} \mathcal{J}^\epsilon_\mathcal{U}$.

Same arg. as yesterday $\implies$ if $z_0 \in \dot\Sigma$ is an inj. pt. of $u_M : \dot\Sigma \to M$ with $u_M(z_0) \in \mathcal{U}$, can pick Y^ξ near $u_M(z_0)$ to cause a contra. unless $\theta^\xi \equiv 0$ near z_0.

$D^\epsilon_{u_0}$ & $D^\xi_{u_0}$ are CR-type ops. on $u_0^* \epsilon$ & $u_0^* \xi$, $D^{\xi\epsilon}_{u_0}$ & $D^{\epsilon\xi}_{u_0}$ are bndl maps.

Assume $\theta^\xi \equiv 0$ near z_0, so $\theta = \begin{pmatrix} \theta^\epsilon \\ 0 \end{pmatrix}$. $\quad \langle D_{u_0}\eta, \theta \rangle_{L^2} = \langle D^\epsilon_{u_0} \eta^\epsilon, \theta^\epsilon \rangle_{L^2} + \langle D^{\epsilon\xi}_{u_0} \eta^\xi, \theta^\epsilon \rangle_{L^2}$

$= 0 \quad \forall\ \eta^\xi, \eta^\epsilon$

$$\implies \begin{cases} \langle D^\epsilon_{u_0} \eta^\epsilon, \theta^\epsilon \rangle_{L^2} = 0 \implies \theta^\epsilon \text{ has isolated zeroes.} \\ \langle D^{\epsilon\xi}_{u_0} \eta^\xi, \theta^\epsilon \rangle_{L^2} = 0. \implies (?) \end{cases}$$

Lemma: $D^{\epsilon\xi}_{u_0} \eta^\xi = -d\lambda(\eta^\xi, du_0 \circ j(\cdot))\, \partial_r + d\lambda(\eta^\xi, du_0(\cdot))\, R$.

pf: Write $\eta^\xi = \partial_\rho u_\rho|_{\rho=0}$ for maps $u_\rho : \dot\Sigma \to \mathbb{R}\times M$,

compute the ϵ-cpnt of $\nabla_\rho \bar\partial_J(u_\rho)|_{\rho=0}$: $dr(\cdot)$ gives ∂_r-cpnt

$\lambda(\cdot)$ gives R-cpnt

Use $d\lambda(v, w) = \partial_v[\lambda(w)] - \partial_w[\lambda(v)] - [v, w]$. $\square$

$$\Rightarrow \begin{cases} \langle D^\epsilon_{u_0}\eta^\epsilon, \theta^\epsilon\rangle_{L^2} = 0 \Rightarrow \theta^\epsilon \text{ has isolated zeroes.} \\ \langle D^{\epsilon\xi}_{u_0}\eta^\xi, \theta^\epsilon\rangle_{L^2} = 0. \Rightarrow (?) \end{cases} = 0 \quad \forall\ \eta^\xi, \eta^\epsilon$$

<u>Lemma</u>: $D^{\epsilon\xi}_{u_0}\eta^\xi = -d\lambda(\eta^\xi, du_0 \circ j(\cdot))\,\partial_r + d\lambda(\eta^\xi, du_0(\cdot))\,R.$

Assume our inj pt. z_0 also satisfies $\operatorname{im} du_0(z_0) \cap \xi^{\perp d\lambda}_{u_0(z)}$.

$\Rightarrow$ in local coords., $d\lambda(\cdot, \partial_t u(z_0))$, $d\lambda(\cdot, \partial_s u_0)\big|_{\xi_{u_0(z)}} \in \operatorname{Hom}(\xi_{u_0(z)}, \mathbb{R})$ are linearly indep. $\Rightarrow D^{\epsilon\xi}_{u_0}$ is a rank 2 bundl map $u_0^*\xi \to \operatorname{Hom}_{\mathbb{C}}(T\dot\Sigma, u_0^*\epsilon)$ near z_0, i.e. it is surjective.

$\Rightarrow$ Can choose η^ϵ near z_0 s.t. $\langle D_0^{\epsilon\xi}\eta^\xi, \theta^\epsilon\rangle_{L^2} > 0$ unless $\theta^\epsilon \equiv 0$ near z_0.

$\Rightarrow \theta \equiv 0$ near z_0, contradiction!

$\square$