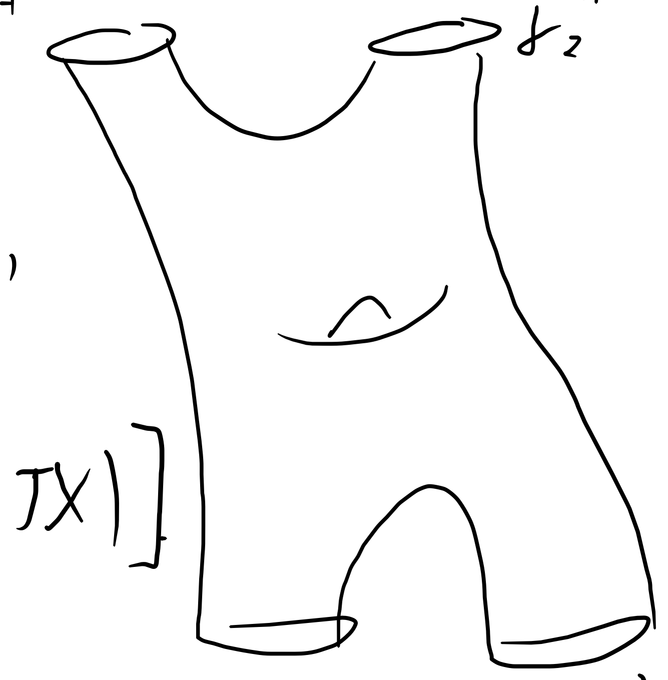
<u>compactness</u>: $\mathcal{M}_{g,m}(A, J, \gamma^+, \gamma^-)$ is not generally compact.

<u>Q</u>: If $u_n \in \mathcal{M}(J)$ has no conv. subseq., what happens instead?

$$E(u) := \sup_{\varphi \in \mathcal{T}} \int_{\dot\Sigma} u^* \omega_\varphi \quad \text{where} \quad \omega_\varphi = \begin{cases} \omega_+ + d(\varphi(r)\lambda_+) & \text{on } [0,\infty) \times M_+ \\ \omega & \text{on } W \\ \omega_- + d(\varphi(r)\lambda_-) & \text{on } (-\infty, 0] \times M_- \end{cases}$$

$$\mathcal{T} := \{ \varphi \in C^\infty(\mathbb{R}, (-\varepsilon, \varepsilon)) \mid \varphi' > 0, \quad \varphi(r) = r \text{ near } r = 0 \}$$

<u>rk</u>: $E(u)$ only depends on γ_+, γ_-, $A \in H_2(W, \bar\gamma^+ \cup \bar\gamma^-)$ & $\omega, \omega_\pm, \lambda_\pm \Rightarrow \forall$ ~~seqs.~~ $u_n \in \mathcal{M}(J) := \mathcal{M}_{g,m}(A, J, \gamma^+, \gamma^-)$, $E(u_n)$ is bdd.



ω_φ tame $J \Rightarrow \exists$ Riem. metric $g(X, Y) := \frac{1}{2}[\omega_\varphi(X, JY) + \omega_\varphi(Y, JX)]$

Then in hol. coords (s,t), $u^*\omega_\varphi(\partial_s, \partial_t) = \omega_\varphi(\partial_s u, \partial_t u)$

$$= \omega_\varphi(\partial_s u, J\partial_s u) = |\partial_s u|^2 = -\omega_\varphi(\partial_t u, \partial_s u) = \omega_\varphi(\partial_t u, J\partial_t u) = |\partial_t u|^2.$$

$\Rightarrow$ If u_n is C^0-bdd, $E(u_n)$ bdd $\Rightarrow u_n$ is $W^{1,2}$-bdd

<u>prop 0</u>: If $J_n \xrightarrow{C^\infty} J$ & $j_n \xrightarrow{C^\infty} j$ & $u_n : (\dot\Sigma, j_n) \longrightarrow (\hat W, J_n)$ satisfies a uniform $W^{1,p}$-bound for some $p > 2$, then u_n has a C^∞_{loc}-conv. subseq. (In particular, C^1-bound suffices.) <u>Pf</u>: Elliptic regularity. $\boxtimes$

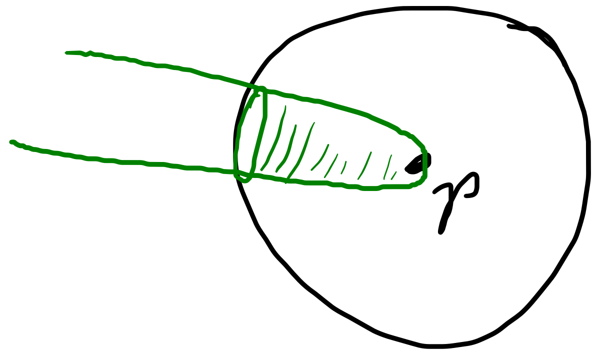
__thm__: Spse $u : (\dot{\Sigma}, j) \longrightarrow (\hat{W}, J)$ is J-hol. & $E(u) < \infty$.
Then if $\nexists$ removable punctures, (+ one more technical assumption)
then u is asymp. cyl. e.g. all Reeb orbits are nondegenerate

__lemma 1__: $u : \dot{\Sigma} \longrightarrow \hat{W}$ is a proper map.

__tools__:

(1) __Gromov's removable singularity thm__: If $u : (\dot{\mathbb{D}}, i) = \mathbb{D} \setminus \{0\} \longrightarrow (\hat{W}, J)$ has precompact image & $\int_{\dot{\mathbb{D}}} u^* \Omega < \infty$ for some symp. form Ω taming J, then u extends smoothly to $\mathbb{D}$.

(2) __monotonicity lemma__: $W^R := ([-R, 0] \times M_-) \cup_{M_-} W \cup_{M_+} ([0, R] \times M_+)$, $\varphi \in \mathcal{T}$, so ω_φ tames $J \in \mathcal{J}(\omega, \mathcal{H}_+, \mathcal{H}_-) \rightsquigarrow$ Riem. metric $g(X, Y) = \frac{1}{2}[\omega_\varphi(X, JY) + \omega_\varphi(Y, JX)]$. Then $\exists$ consts. $c, r_0 > 0$ s.t. $\forall$ balls $B_r(p) \subseteq W^R$ with $r < r_0$, every proper nonconst. J-hol. map $u : (\Sigma, j) \longrightarrow B_r(p)$, passing through p, satisfies $\int_\Sigma u^* \omega_\varphi \geq c r^2$.
$(\partial \Sigma = \emptyset)$



Lemma 2: Suppose $\mathcal{H} = (\omega, \lambda)$ a SHS on M, $J \in \mathcal{J}(\mathcal{H})$, $u : \dot{\Sigma} \longrightarrow \mathbb{R} \times M$ J-hol. w/ $E(u) < \infty$, $\int_{\dot{\Sigma}} u^* \omega = 0$.

(a) If $\dot{\Sigma} = \mathbb{R} \times S^1$, then u is either a trivial cylinder over a Reeb orbit or is constant.

(b) If $\dot{\Sigma} = \mathbb{C}$, then u is constant.

Pf: $\omega|_\xi$ tames $J|_\xi$ & $\omega(R, \cdot) \equiv 0$ $\Rightarrow$ in hol. coords. (s,t),

$$u^* \omega(\partial_s, \partial_t) = \omega(\partial_s u, \partial_t u) = \omega(\pi_\xi \partial_s u, \pi_\xi \partial_t u) = \omega(\pi_\xi \partial_s u, J \pi_\xi \partial_s u)$$

≥ 0, $=$ iff $\pi_\xi \partial_s u = 0$, $\Rightarrow$ $\pi_\xi \partial_t u = 0$ $\Rightarrow$ u is everywhere tangent to the J-inv. foliation spanned by ∂_r & R. $\Rightarrow$ $\operatorname{im} u \subseteq \mathbb{R} \times \gamma(\mathbb{R})$ for some Reeb orbit $\gamma : \mathbb{R} \longrightarrow M$.

If γ not closed: u factors through the J-hol. map $v : \mathbb{C} \longrightarrow \mathbb{R} \times M$

$$s + it \longmapsto (s, \gamma(t)).$$

EX: $E(v) = \sup_{\varphi \in \mathcal{T}} \int_{\mathbb{C}} v^* d(\varphi(r)\lambda) = \infty \Rightarrow E(u) = \infty$ unless u is constant.

If γ is closed: u factors through $u_\gamma : \mathbb{R} \times S^1 \longrightarrow \mathbb{R} \times M : (s,t) \longmapsto (Ts, \gamma(t))$ for $T :=$ minimal period of γ. Say $u = u_\gamma \circ \varphi$, $\varphi : \dot{\Sigma} \longrightarrow \mathbb{R} \times S^1$.

If $\dot{\Sigma} = \mathbb{C}$, $\varphi : \mathbb{C} \longrightarrow \mathbb{R} \times S^1 \cong \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ hol. $\Rightarrow$ φ has an essential sing. at ∞ $\overset{(EX)}{\Longrightarrow}$ $E(u) = \infty$.

If $\dot{\Sigma} = \mathbb{R} \times S^1$ & $u \neq$ const., $\varphi =$ finite cover $\varphi(s,t) = (ks, kt)$ (up to reparametrization) $\Rightarrow$ u is a trivial cylinder. $\square$

<u>Lemma 3</u>: If $u : Z_+ \overset{[0,\infty)\times S^1}{:=} \longrightarrow \mathbb{R}\times M$ is J-hol. & $E(u) < \infty$, then

$\exists$ bound $|du(s,t)| \leq C$ w.r.t. any $\mathbb{R}$-inv't metric.

pf in a moment.

Now, given $u = (u^{\mathbb{R}}, u^M) : Z_+ \longrightarrow \mathbb{R}\times M$ w/ $E(u) < \infty$, let

$$u_k : [-k,\infty)\times S^1 \longrightarrow \mathbb{R}\times M : (s,t) \longmapsto \left(u^{\mathbb{R}}(s+s_k,t) - u^{\mathbb{R}}(s_k,0),\ u^M(s+s_k,t)\right)$$

for a seq. $s_k \longrightarrow \infty$. $\quad u_k(0,0) = (0, u^M(s_k,0)) \in \{0\}\times M \overset{\text{cpct}}{\subseteq} \mathbb{R}\times M$,

& $|du_k|$ unif bdd $\Longrightarrow$ u_k is unif C^1-bdd on cpct subsets

$\Longrightarrow$ a subseq. conv. in $C^\infty_{loc}(\mathbb{R}\times S^1)$ to a J-hol. cylinder

$u_\infty : \mathbb{R}\times S^1 \longrightarrow \mathbb{R}\times M$, $\quad E(u_\infty) < \infty$.

$$\left.\begin{aligned} \int_{[-s_k,0]\times S^1} u_k^*\omega &= \int_{[0,\infty)\times S^1} u^*\omega \\ \int_{[-s_k/2,\infty)\times S^1} u_k^*\omega &= \int_{[\frac{s_k}{2},\infty)\times S^1} u^*\omega \xrightarrow{k\to\infty} 0 \end{aligned}\right\} \Longrightarrow \int_{\mathbb{R}\times S^1} u_\infty^*\omega = 0.$$

$\overset{\text{Lemma 2}}{\Longrightarrow}$ u_∞ is a trivial cylinder.

<u>pf of Lemma 3</u>: ("bubbling off") Spse $\exists$ a seq. $z_k \in Z_+$ s.t.

$R_k := |du(z_k)| \longrightarrow \infty$. Pick $\varepsilon_k > 0$ s.t. $\varepsilon_k \longrightarrow 0$, $\varepsilon_k R_k \longrightarrow \infty$,

consider $v_k : (\mathbb{D}_{\varepsilon_k R_k}, i) \longrightarrow (\mathbb{R}\times M, J)$, $\quad v_k(z) := \tau_k \circ u\left(z_k + \frac{z}{R_k}\right)$

where $\tau_k : \mathbb{R}\times M \longrightarrow \mathbb{R}\times M : (r,x) \longmapsto (r + c_k, x)$ s.t. $v_k(0) \in \{0\}\times M$.

Now $|dv_k(z)| = \frac{1}{R_k}\left|du\left(z_k + \frac{z}{R_k}\right)\right|$, $\Rightarrow$ $|dv_k(0)| = 1$. $\forall\ k$.

<u>Hofer lemma</u>: (X,d) a complete metric space, $g : X \longrightarrow [0,\infty)$ contin.,
$x_0 \in X$, $\varepsilon_0 > 0$. Then $\exists$ $x \in X$ & $\varepsilon > 0$ s.t.

(i) $\varepsilon \le \varepsilon_0$, (ii) $\varepsilon\, g(x) \ge \varepsilon_0\, g(x_0)$, (iii) $d(x, x_0) \le 2\varepsilon_0$,
(iv) $g(y) \le 2 g(x)$ $\forall\ y \in \overline{B_\varepsilon(x)}$. Pf: Analysis I.

Now WLOG, $|du(z)| \le 2|du(z_k)|$ $\forall\ z \in \overline{B_{\varepsilon_k}(z_k)}$

$\Rightarrow$ $|dv_k(z)| \le 2$ $\forall\ z \in \mathbb{D}_{\varepsilon_k R_k}$ $\Rightarrow$ v_k is unif. C^1-bdd on cpt subsets.

Prop. 0
$\Rightarrow$ v_k (subseq) $\xrightarrow{C^\infty_{loc}}$ $v_\infty : \mathbb{C} \longrightarrow \mathbb{R}\times M$ J-hol., $E(v_k)$ unif bdd $\Rightarrow$
$E(v_\infty) < \infty$, $\displaystyle\int_{\mathbb{C}} v_\infty^* \omega = 0$ $\overset{\text{Lemma 2}}{\Longrightarrow}$ v_∞ is constant
but $|dv_\infty(0)| = 1$

$\Rightarrow$ contradiction! $\square$