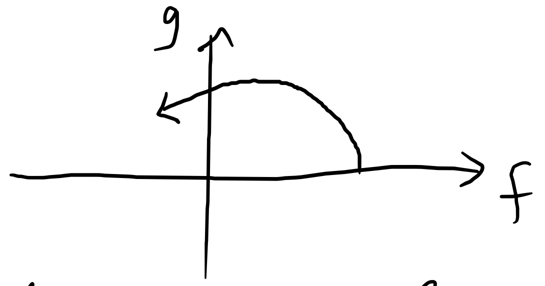
# application in contact topology

ex 0: $\mathbb{R} \times \mathbb{T}^2$, $\rho \in \mathbb{R}$, $(\phi, \theta) \in \mathbb{T}^2$, $\alpha := f(\rho)\, d\theta + g(\rho)\, d\phi \implies \alpha \wedge d\alpha = D(\rho)\, d\theta \wedge d\rho \wedge d\phi$

where $D := fg' - f'g$, $D > 0 \iff \alpha$ is a pos. ctt form




Let $\alpha_{GT} := \cos(2\pi\rho)\, d\theta + \sin(2\pi\rho)\, d\phi$

$\xi_{GT} := \ker \alpha_{GT}$.

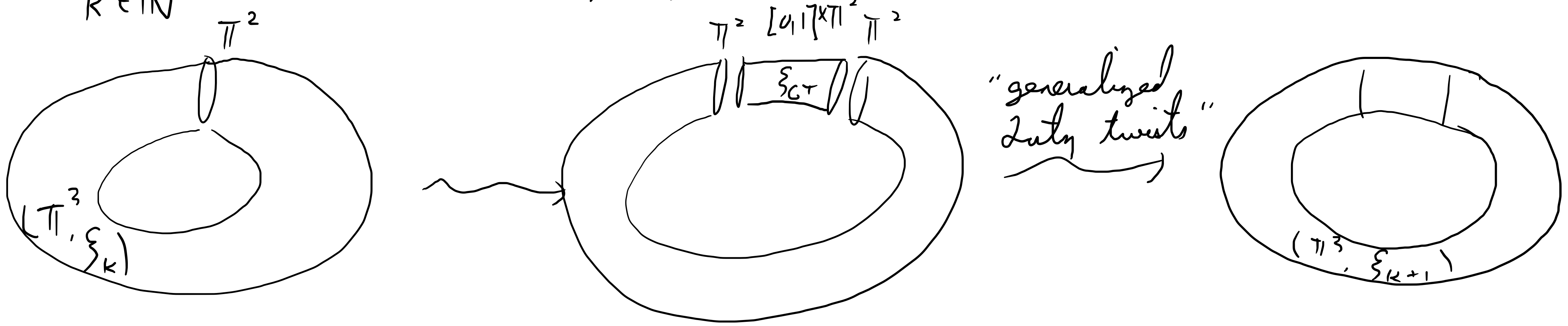
defn: The Giroux torsion of a ctt 3-mfd $(M, \xi)$ is

$$GT(M, \xi) := \sup\{ k \geq 0 \text{ integers} \mid \exists \text{ ctt embedding } ([0,k] \times \mathbb{T}^2, \xi_{GT}) \hookrightarrow (M, \xi) \}$$

$\in \mathbb{N} \cup \{0, \infty\}$

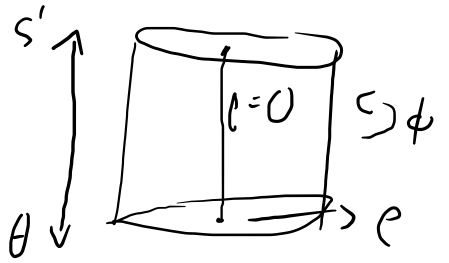
ex 1: $(\mathbb{T}^3, \xi_k) := (\mathbb{R} \times \mathbb{T}^2, \xi_{GT}) / k\mathbb{Z}$ has $GT(\mathbb{T}^3, \xi_k) \geq k - 1$.

$k \in \mathbb{N}$




prop: Lutz twist leaves htpy class of the ctt str. unchanged. $\square$

ex 2: $k \geq 0$, $(S^1 \times S^2, \xi_k) := ([0, k + \frac{1}{2}] \times \mathbb{T}^2, \xi_{GT}) / (\rho, \phi, \theta) \sim (\rho, \phi', \theta)$ for $\rho = 0$ or $\rho = k + \frac{1}{2}$




glued to another copy itself $\rightsquigarrow S^2 \times S^1$

prop (using Eliashberg's "flexibility" thm): Since for $k \geq 1$, $(S^1 \times S^2, \xi_k)$ are all overtwisted, they are all contactomorphic. (not for $k = 0$)

thm: $(\mathbb{T}^3, \xi_k)$ & $(\mathbb{T}^3, \xi_l)$ are contactomorphic iff $k = l$.

# cylindrical cont homology

$(M^{2n-1}, \xi = \ker\alpha)$ cont mfd, $J \in \mathcal{J}(\alpha) := \mathcal{J}((d\alpha, \alpha))$

prop 1: all nonconst. asymp. cyl. $u : (\dot\Sigma, j) \longrightarrow (\mathbb{R}\times M, J)$ have $\geq 1$ pos. puncture.


pf 1: $0 < E(u) = \sup_{\varphi\in\mathcal{T}} \int_{\dot\Sigma} u^* d\left((1+\varphi)\alpha\right)$

$$= \sup_{\varphi\in\mathcal{T}}\left(\sum_{z\in\Gamma^+}[1+\varphi(\infty)]\underbrace{T(\gamma_z)}_{\text{period}} - \sum_{z\in\Gamma^-}[1+\varphi(-\infty)]T(\gamma_z)\right).$$ ☐

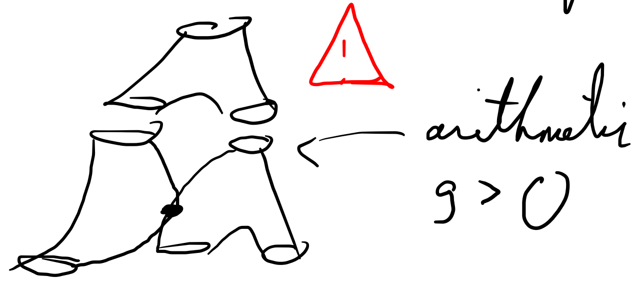
rk: $E(u)$ is bdd above in terms of periods of pos. asymp. orbits.

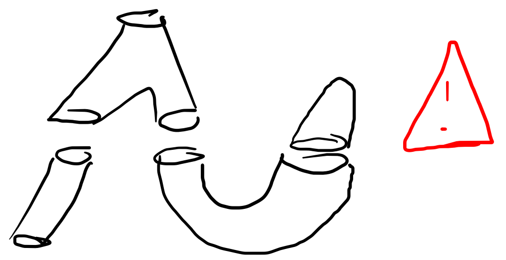
pf 2: CR-eqn $\iff$ $u = (u_{\mathbb{R}}, u_M) : \dot\Sigma \longrightarrow \mathbb{R}\times M$ satisfies
$$\begin{cases} \partial_s u_{\mathbb{R}} - \alpha(\partial_t u_M) = 0 \\ \partial_t u_{\mathbb{R}} + \alpha(\partial_s u_M) = 0 \\ \pi_\xi \partial_s u_M + J\pi_\xi \partial_t u_M = 0 \end{cases}$$

$\Rightarrow \Delta u_{\mathbb{R}} - d\alpha(\partial_s u_M, \partial_t u_M) = 0$

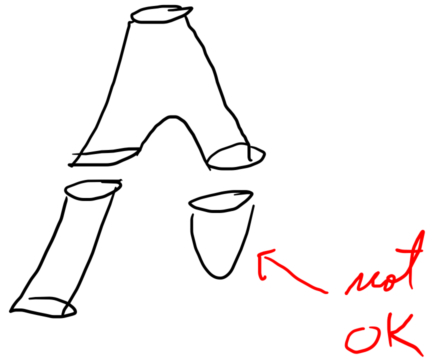
$\Rightarrow -\Delta u_{\mathbb{R}} = -d\alpha(\pi_\xi \partial_s u_M, J\pi_\xi\partial_s u_M) \leq 0 \overset{\text{(max. principle)}}{\Longrightarrow} u_{\mathbb{R}}$ has no local maxima. ☐

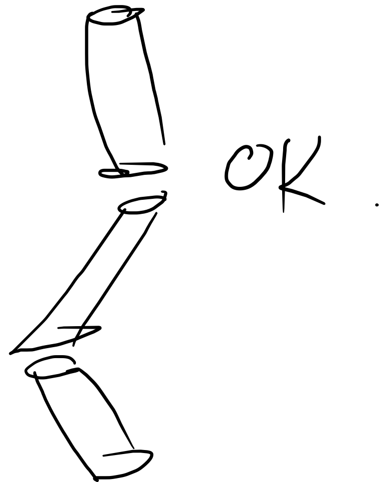
EX: Given Prop. 1, if $u \in \overline{\mathcal{M}}(J)$ is a hol. building w/ $g = 0$ & 1 pos. pct., then all cpts of $u$ also have those two properties, & $\nexists$ nodes.

cor: If $u \in \overline{\mathcal{M}}(J)$ has $\#\Gamma^\pm = 1$, $g = 0$ & $\nexists$ planes, then every level of $u$ is a single hol. cylinder w/ a pos. & neg end.



Assume $h \in [S^1, M]$ is primitive.

Our cont form $\alpha$ is admissible if (i) all orbits $\mathcal{P}_h(\alpha) := \left\{ \begin{array}{l} \text{closed Reeb orbits} \\ \gamma,\ [\gamma] = h \end{array} \right\}$ are nondegenerate, (ii) $\nexists$ contractible Reeb orbits.

$$CC_*^h(M,\alpha) := \bigoplus_{\gamma \in \mathcal{P}_h(\alpha)} \mathbb{Z}_2 \quad \text{with } \mathbb{Z}_2\text{-grading by } \mu_{CZ}(\gamma) + (n-3) \in \mathbb{Z}_2.$$

$$\partial : CC_*^h(M,\alpha) \longrightarrow CC_{*-1}^h(M,\alpha),$$

$$\partial\langle\gamma\rangle := \sum_{\gamma'} \left( \#_{\mathbb{Z}_2} \left( \begin{array}{l} u : \mathbb{R}\times S^1 \longrightarrow \mathbb{R}\times M \\ J\text{-hol. of index } 1 \end{array} \right) \Big/ \mathbb{R} \right) \langle\gamma'\rangle.$$

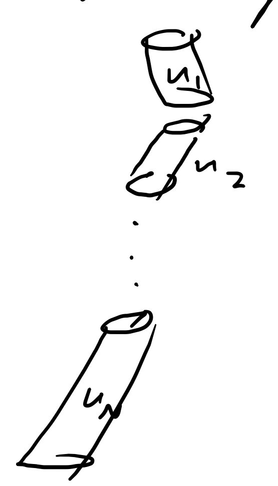
rk 1: $J$ generic; since $h$ is primitive, all orbits in $\mathcal{P}_h(\alpha)$ are simply covered $\Rightarrow$ all $J$-hol. cyls. are somewhere inj. $\Rightarrow$ all Fredholm regular.

rk 2: For , $\operatorname{ind}(u) = (n-3)\underbrace{\chi(\mathbb{R}\times S^1)}_{=0} + 2c_1^\tau(u^*T(\mathbb{R}\times M)) + \mu_{CZ}^\tau(\gamma) - \mu_{CZ}^\tau(\gamma')$

$= 1 \implies \mu_{CZ}^\tau(\gamma) - \mu_{CZ}^\tau(\gamma')$ is odd $\implies \partial$ is odd.

rk 3: for $\gamma, \gamma' \in \mathcal{P}_h(\alpha)$ not the same $\Rightarrow \operatorname{ind} \geq 1$ (by regularity).

$\Rightarrow$ for any $J$-hol. building w/ $g = 0$, $\#\Gamma^\pm = 1$ & $N \geq 1$ levels,

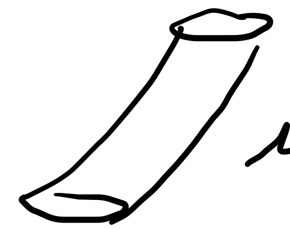


$$\Rightarrow \sum_{i=1}^{N} \operatorname{ind}(u_i) \geq N.$$

$\parallel$

index of any seq. of smooth $J$-hol. cylinders converging to the building.

**lemma 1:** $\partial^2 = 0$.

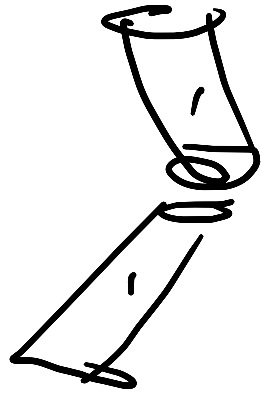
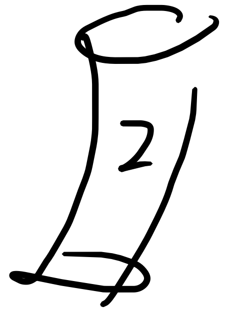
**pf:** Let $\mathcal{M}_k := \{$  index $k \}/\mathbb{R}$.

$\mathcal{M}_2$ = 1-mfd

$\Rightarrow$ $\overline{\mathcal{M}}_2$ = cpct 1-mfd w/ bdry $\partial \overline{\mathcal{M}}_2 = \{$  $\}$

counted by $\partial^2$.

$\Rightarrow$ count is even. $\blacksquare$

(rk: This also requires gluing, i.e. for any broken cyl. , it is the limit of a unique $[0,\infty)$-parametrized family of cyls. .)

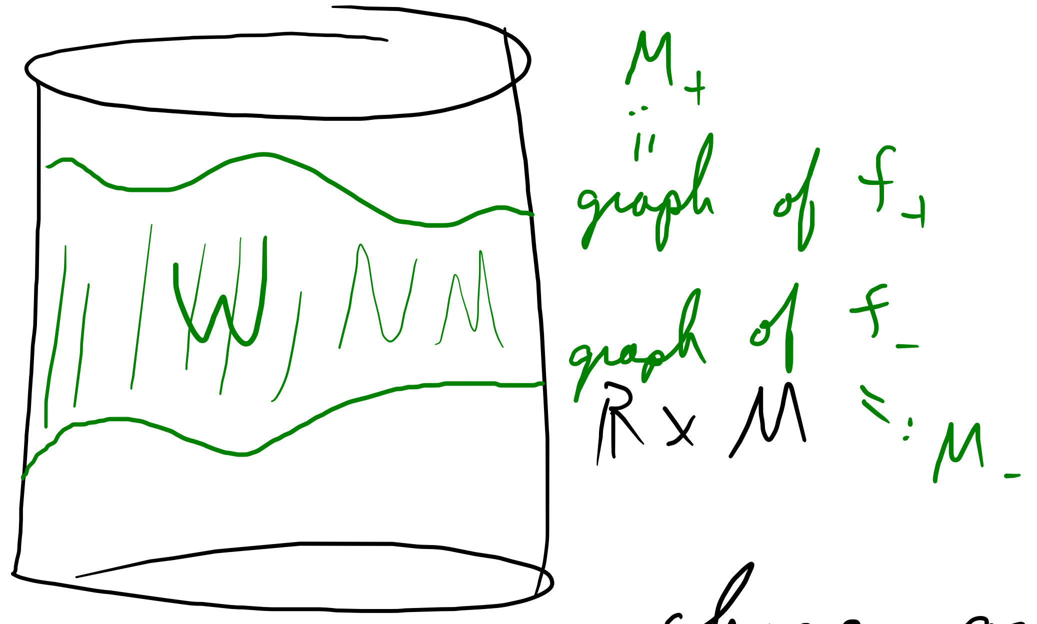
**defn:** $HC^h_*(M,\alpha,J) := H_*(CC^h_*(M,\alpha), \partial)$.

invariance: Assume $\alpha_\pm$ both admissible cont. forms for $\xi_i$ rescale s.t.

$\alpha_+ > \alpha_-$ (meaning $\alpha_+ = f\alpha_-$ for a fn. $f > 1$).

Write $\alpha_\pm = e^{f_\pm}\alpha$ for some cont. form $\alpha$ & fn. $f_\pm : M \longrightarrow \mathbb{R}$.

$f_+ > f_-$

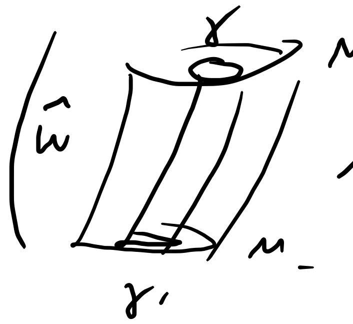



$W :=$ exact symp. cobordism w/ Liouville form

$\lambda = e^r\alpha$, so $\lambda|_{M_\pm} = \alpha_\pm$.

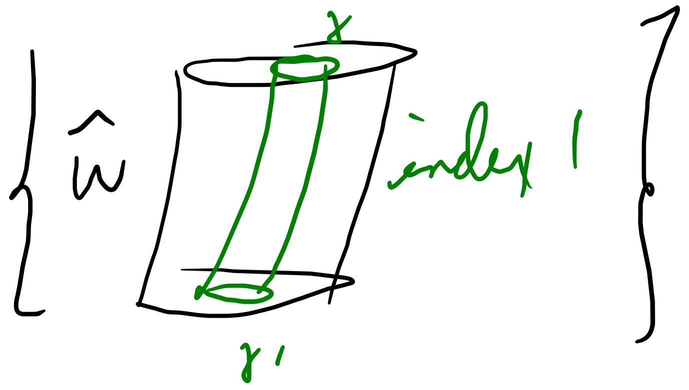
⇝ completion $\hat{W}$,

choose generic $J \in \mathcal{J}(d\lambda, \alpha_+, \alpha_-)$ (i.e. $J|_{[0,\infty)\times M_+} \in \mathcal{J}(\alpha_+)$,
$J|_{(-\infty,0]\times M_-} \in \mathcal{J}(\alpha_-)$).

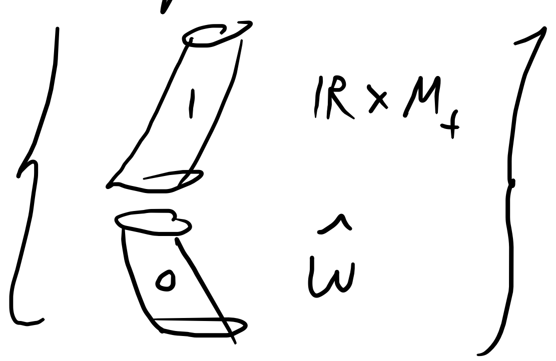
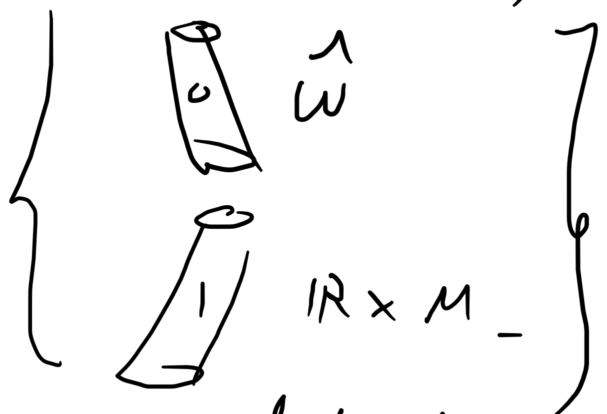
$\Phi_J : CC_*^h(M,\alpha_+) \longrightarrow CC_*^h(M,\alpha_-) : \langle\gamma\rangle \longmapsto \sum_{\gamma' \in \mathcal{P}_h(\alpha_-)} \#_{\mathbb{Z}_2}\Big($  $\Big)\langle\gamma'\rangle$.



Lemma 2: $\Phi_J \circ \partial_+ = \partial_- \circ \Phi_J$.

Pf: $\Big\{$  $\Big\}$ is a 1-dim. mfd w/ compactification having



bdry $\Big\{$  $\Big\}$ $\sqcup$ $\Big\{$  $\Big\}$





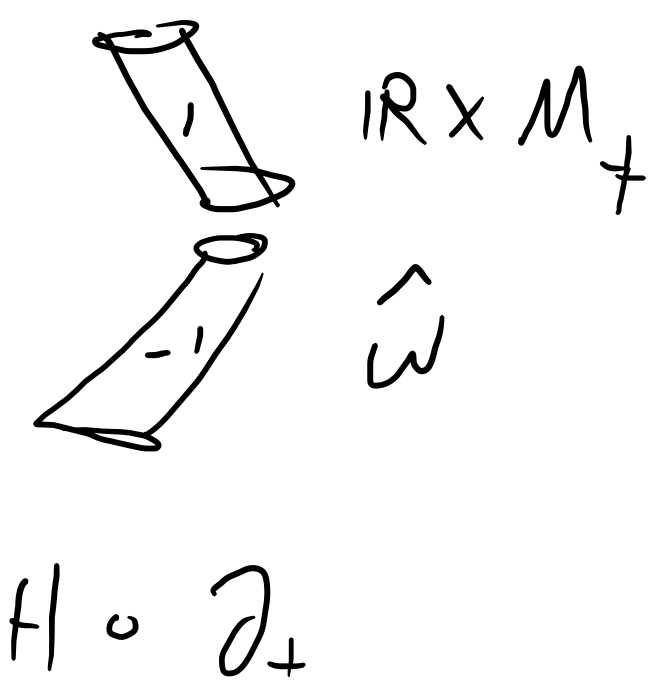
counted by $\Phi_J \circ \partial_+$      counted by $\partial_- \circ \Phi_J$ ∎

**Lemma 3**: Induced map $HC_*^h(M, \alpha_+, J_+) \longrightarrow HC_*^h(M, \alpha_-, J_-)$ is indep. of $J$ (matching $J_\pm \in \mathcal{J}(\alpha_\pm)$ on ends).

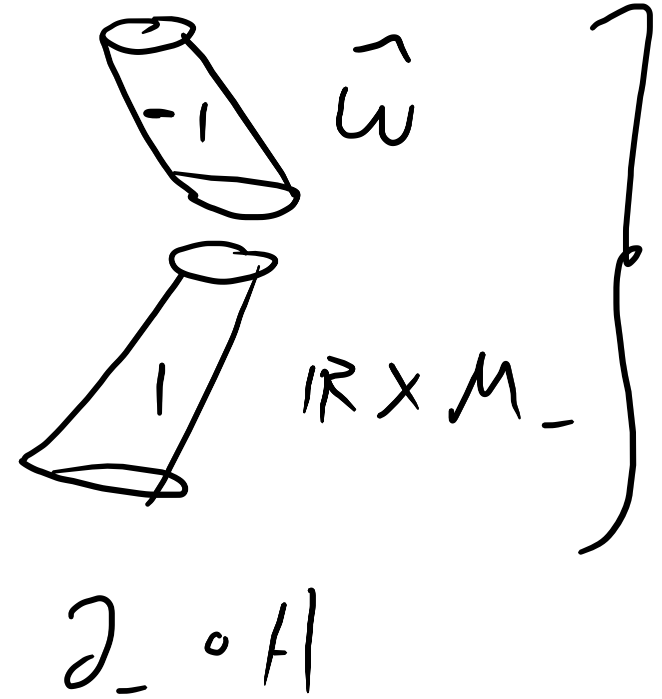
Pf: Given 2 choice $J_0, J_1$ on $\hat{W}$, choose a generic htpy $\{J_s\}_{s\in[0,1]}$,

$\rightsquigarrow \mathcal{M}_k(\{J_s\}) = \{(s,u) \mid s \in [0,1],\ u \in \mathcal{M}_k(J_s)\}$ is a mfd of dim. $k+1$. Consider $\mathcal{M}_0(\{J_s\})$:

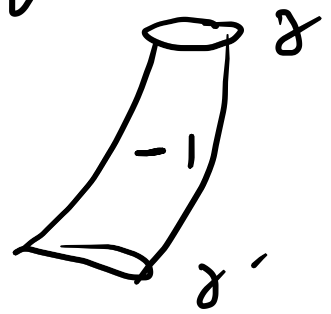
$\partial \overline{\mathcal{M}_0(\{J_s\})} = \underset{\Phi'_{J_0}}{\mathcal{M}_0(J_0)} \amalg \underset{\Phi'_{J_1}}{\mathcal{M}_0(J_1)} \amalg$




$\amalg$




for $H : CC_*(M, \alpha_+) \longrightarrow CC_{*+1}(M, \alpha_-)$: def'd by counting pairs $(s,u)$ s.t. $s \in [0,1]$ & $u =$



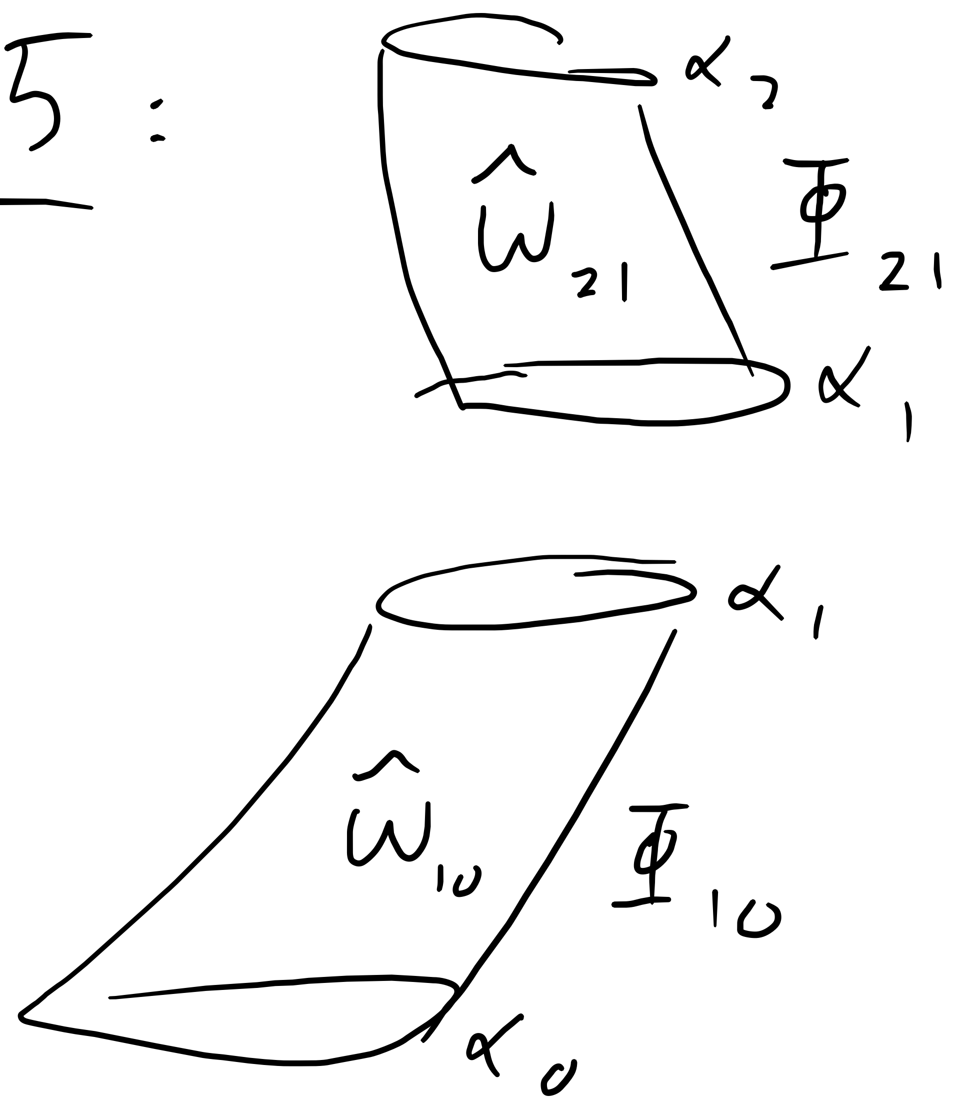

$J_s$-hol.

$\Rightarrow H$ is a chain htpy btwn $\Phi_{J_0}$ & $\Phi_{J_1}$.

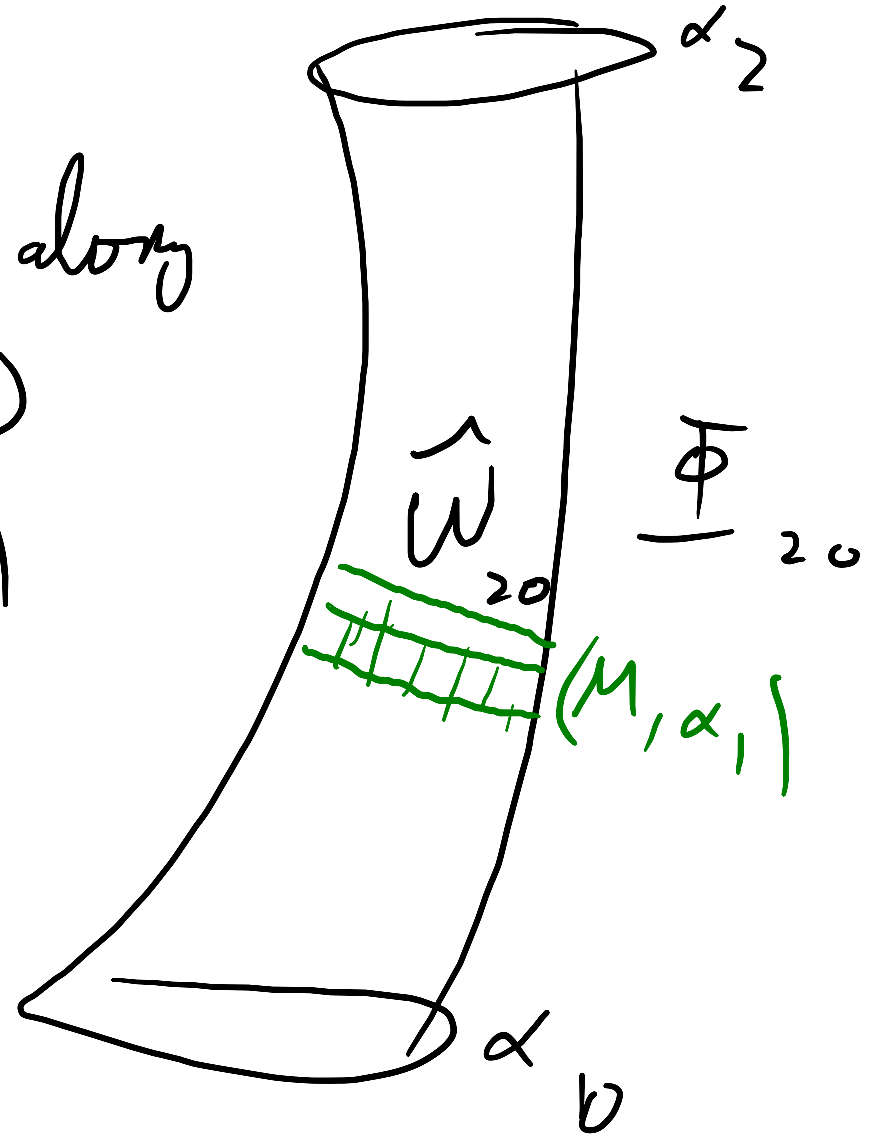
**Lemma 4**: For $c = \text{const} < 1$, $HC_*(M, \alpha, J) \xrightarrow{\Phi_J} HC_*(M, c\alpha, J)$ is Id.

Pf: Count $J$-hol. trivial cyls. $\square$

**Lemma 5**:




attach along $(M, \alpha_1)$




Then $\Phi_{20} = \Phi_{21} \circ \Phi_{10}$.

Pf: Stretching the neck. $\square$