computation of $HC_*^h(\mathbb{T}^3, \xi_k)$

$\left(\mathbb{T}^3 = S^1 \times S^1 \times S^1 \ni (\rho, \phi, \theta)\right)$

<u>thm</u>: For $k \in \mathbb{N}$ & a primitive htpy class $h \in [S^1, \mathbb{T}^3]$,

(1) If h has a nontrivial image under proj. $\mathbb{T}^3 \to S^1 : (\rho, \phi, \theta) \mapsto \rho$,

then $HC_*^h(\mathbb{T}^3, \xi_k) = 0$.

(2) Otherwise, $HC_*^h(\mathbb{T}^3, \xi_k) = \begin{cases} \mathbb{Z}_2^k & \text{if } * = \text{odd} \\ \mathbb{Z}_2^k & \text{if } * = \text{even}. \end{cases}$

<u>cor</u>: $(\mathbb{T}^3, \xi_k) \not\cong (\mathbb{T}^3, \xi_\ell)$ if $k \neq \ell$.

<u>Recall</u>: $\xi_k = \ker \alpha_k$, $\alpha_k := \cos(2\pi k\rho)\, d\theta + \sin(2\pi k \rho)\, d\phi$.

$\rightsquigarrow$ Reeb fld $R_{\alpha_k} = \cos(2\pi k \rho)\, \partial_\theta - \sin(2\pi k\rho)\, \partial_\phi$.

If h is nontrivial in ρ-coord., $\mathcal{P}_h(\alpha_k) = \emptyset \implies HC_*^h(\mathbb{T}^3, \xi_k) = 0$.

<u>lemma</u>: For any $h \in [S^1, \mathbb{T}^3]$ that is trivial in ρ-coord., $\exists$ a contactomorphism $(\mathbb{T}^3, \xi_k) \circlearrowleft$ identifying h with $[t \mapsto (0, 0, t)]$. $\blacksquare$

Fix this h from now on, consider case $k = 1$.

(General case will follow by covering.)

$\alpha = \cos(2\pi\rho)\, d\theta + \sin(2\pi\rho)\, d\phi$, $\quad R_\alpha = \cos(2\pi\rho)\,\partial_\theta - \sin(2\pi\rho)\,\partial_\phi$.

<u>recall</u>: α is admissible if $\nexists$ contractible orbits (ok), all orbits in $\mathcal{P}_\lambda(\alpha)$ nondeg.

(not true here: $\{\rho = 0\} \cong \mathbb{T}^2 \subseteq \mathbb{T}^3$ is foliated by an S^1-fam. of orbits in $\mathcal{P}_\lambda(\alpha)$). Need to perturb α.

<u>rk</u>: $\alpha = \cos(2\pi\rho)(d\theta + \beta)$ where $\beta := \tan(2\pi\rho)\, d\phi$ is a Liouville form on $\mathbb{A} := [-\frac{1}{8}, \frac{1}{8}] \times S^1 \ni (\rho, \phi)$.

<u>idea</u>: Recast CH on $(\mathbb{A} \times S^1, \ker(d\theta + \beta)) \subseteq (\mathbb{T}^3, \xi_1)$ as a small perturbation of FH of $(\mathbb{A}, d\beta)$.

choice of Ham. fn. $H : \mathbb{A} \times S^1 \to \mathbb{R}$

<u>Recall</u>: $\exists$ SHS $\mathcal{H}_0 = (\omega := d\beta + d\theta \wedge dH,\ \lambda_0 := d\theta)$ on $\mathbb{A} \times S^1$

s.t. for $J_0 \in \mathcal{J}(\mathcal{H}_0)$ $(\Leftarrow \{J_\theta \in \mathcal{J}(\mathbb{A}, d\beta)\}_{\theta \in S^1})$,

$v : \mathbb{R} \times S^1 \to \mathbb{A}$ satisfies Floer eqn. $\Longleftrightarrow u : \mathbb{R} \times S^1 \to \mathbb{R} \times (\mathbb{A} \times S^1)$

def'd $u(s,t) = (s, v(s,t), t)$ is J_0-hol.

<u>lemma</u>: $\forall\ \varepsilon > 0$ suff. small, $\lambda_\varepsilon := d\theta + \varepsilon(\beta - H\, d\theta) = (1 - \varepsilon H)\, d\theta + \varepsilon\beta$

is ctct on $\mathbb{A} \times S^1$ & has Reeb vec. fld R_ε colinear w/ that of $\mathcal{H}_0$.

($\rightsquigarrow$ smooth family of SHS's $\mathcal{H}_\varepsilon := (\omega, \lambda_\varepsilon)$ for $\varepsilon \geq 0$ small).

<u>Pf</u>: $\lambda_\varepsilon \wedge d\lambda_\varepsilon = \varepsilon \underbrace{d\theta \wedge d\beta}_{>0} + \mathcal{O}(\varepsilon^2) > 0$. $\quad d\lambda_\varepsilon = \varepsilon(d\beta + d\theta \wedge dH) = \varepsilon\omega$

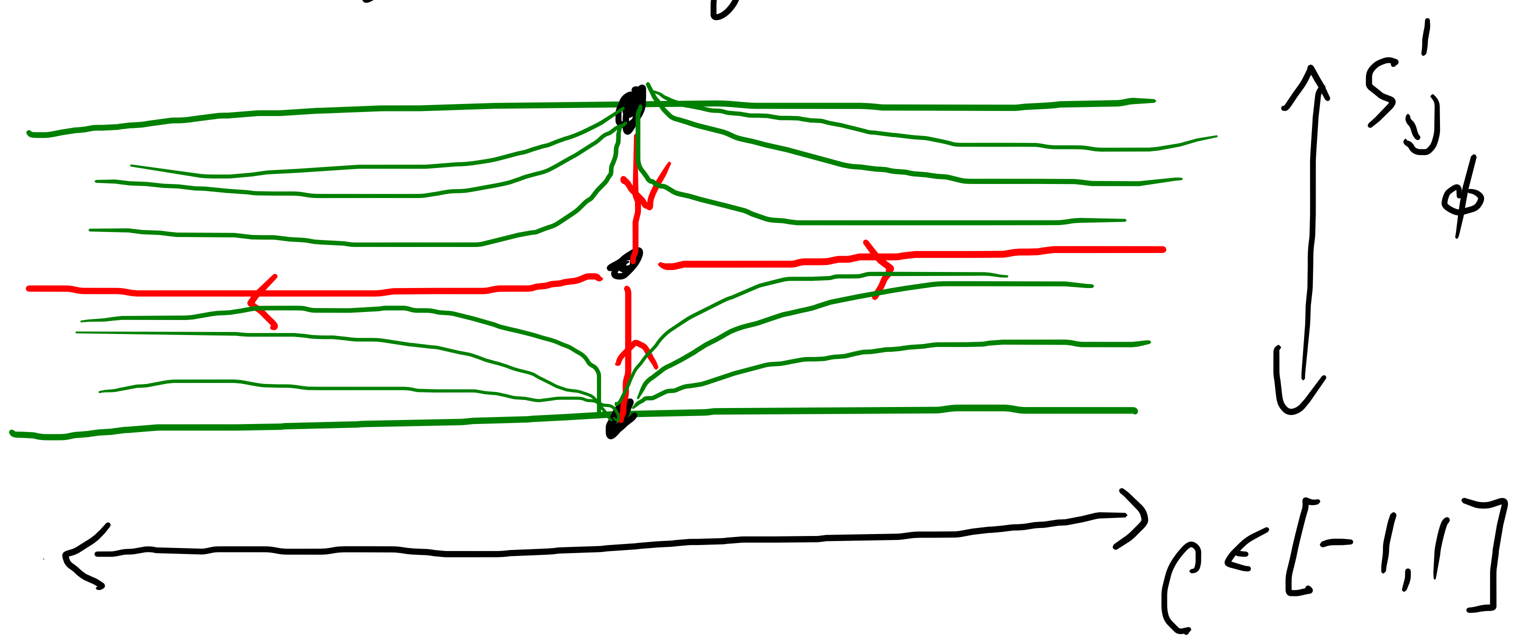
$\Rightarrow \ker d\lambda_\varepsilon = \ker \omega \Rightarrow R_\varepsilon$ & R_0 are colinear. $\square$

<u>Notice</u>: $\mathcal{J}(\mathcal{H}_\varepsilon) = \mathcal{J}(\lambda_\varepsilon)$ for $\varepsilon > 0$ small.

$\Rightarrow$ can consider a.c.s.'s that are small perts. of $J_0 \in \mathcal{J}(\mathcal{H}_0)$ to defn CH chain cpx.

<u>cor</u> (by deforming H): λ_ε is htpic through a family of ctct forms to $d\theta + \varepsilon\beta \Rightarrow$ (by deforming ε) also htpic to $d\theta + \beta$. $\square$

<u>lemma</u> (from Floer hom.): Assume $H : A \longrightarrow \mathbb{R}$ is Morse, $\nexists$ crit. pts. on ∂A, $J \in \mathcal{J}(A, d\beta)$ s.t. ∇-flow of H w.r.t. $g := d\beta(\cdot, J\cdot)$ is Morse-Smale.



Then after replacing H with cH for some $0 < c << 1$, the SHS $\mathcal{H}_0 = (\omega, \lambda_0)$ has the following properties:

(1) $\forall\ x \in \mathrm{Crit}(H)$, $\gamma_x(t) := (x, t) \in A \times S^1$ is a nondeg. 1-periodic orbit with $\mu^\tau_{CZ}(\gamma_x) = 1 - \mathrm{Morse}(x)$ ($\tau = S^1$-inv't triv.)

(2) ∇-flow lines $\gamma(s) \in A \rightsquigarrow$ Fredholm regular sols. $v : \mathbb{R} \times S^1 \longrightarrow A$ to

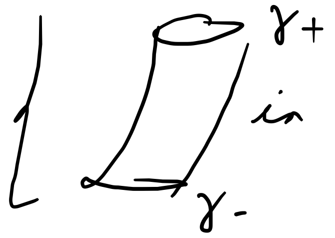
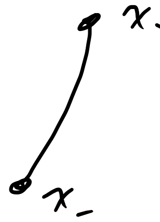
$$(F) \quad \partial_s v + J(v)\big(\partial_t v - X_H(v)\big) = 0 \qquad \text{of form } v(s,t) = \gamma(s).$$

(3) All 1-periodic orbits are as in (1), all sols. to (F) w/ finite energy are as in (2).

cor: If H is suff. small, then $\forall\ \varepsilon > 0$ suff. small,

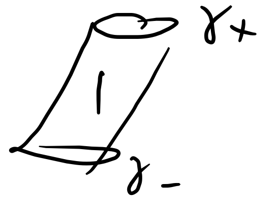
$\exists$ bijection $\mathcal{P}_h(\lambda_\varepsilon) \xleftrightarrow{1:1} \mathrm{Crit}(H) = 2$ pts. $\{x_+, x_-\}$,

$(h := [t \mapsto (\text{const}, t) \in \mathbb{A}\times S^1])$

$\Bigg\{$  in $\mathbb{R}\times(\mathbb{A}\times S^1)\Bigg\} \xleftrightarrow{1:1} \Bigg\{\nabla$-flow line  in $\mathbb{A}\Bigg\}$.

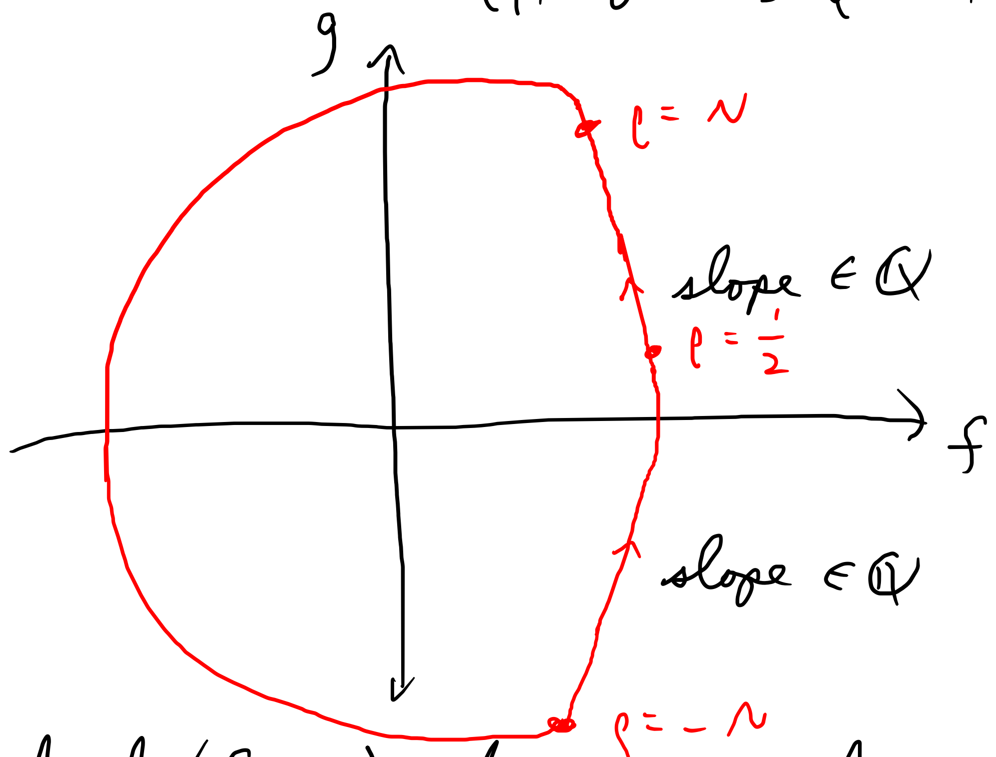
pf: Use IFT to deform from H_0 to H_ε a $J_0 \in \mathcal{J}(H_0)$ to $J_\varepsilon \in \mathcal{J}(H_\varepsilon)$. $\square$

Lemma: One can embed $(\mathbb{A}\times S^1, \ker\lambda_\varepsilon)$ into $(\mathbb{T}^3, \xi_1)$ as a nbhd of $\{\rho = 0\}$

s.t. all all J-hol. cyls.  are contained in $\mathbb{R}\times(\mathbb{A}\times S^1)$.

pf sketch: $\lambda_\varepsilon = (1-\varepsilon H)\,d\theta + \varepsilon\rho\, d\phi$, WLOG $H = H(\rho)$ away from $\{\rho = 0\}$.

$= f(\rho)\,d\theta + g(\rho)\,d\phi$ away from $\{\rho = 0\}$.



WLOG assume $\frac{g'}{f'} = \text{const} \in \mathbb{Q}$

for $|\rho| \geq \frac{1}{2}$

Recall: Reeb fld of $f\,d\theta + g\,d\phi$ is

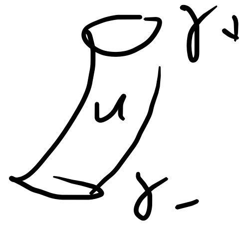
$\frac{g'}{\Delta}\partial_\theta - \frac{f'}{\Delta}\partial_\phi$ for $\Delta := fg' - f'g$

$\Longrightarrow$ all orbits in $|\rho| \geq \frac{1}{2}$ are all periodic.

Extend (f, g) to a periodic path $[-N-1, N+1] \longrightarrow \mathbb{R}^2$ for some $N \gg 0$ s.t. slope in $\{\frac{1}{2} \leq |\rho| \leq N\}$ remains constant & winding $= 1$

$\rightsquigarrow$ ctct form in $\mathbb{T}^3 \cong \mathbb{R}/(2N+2)\mathbb{Z} \times \mathbb{T}^2$ w/ ctct str. isotopic to ξ_1.

remains to show: all J-hol. cyls. in this model connecting the 2 orbits arising from crit. pts. of H <u>stay</u> in the region $\{|\rho| < \frac{1}{2}\}$.

<u>idea</u>: The region $\mathbb{R} \times \{\frac{1}{2} \le |\rho| \le N\} \subseteq \mathbb{R} \times \mathbb{T}^3$ is foliated by J-hol. trivial cyls. <u>claim</u>: If a cyl. 

intersects one of these triv. cyls., then it intersects <u>all</u> of them.

Intersections of u w/ a triv. cyl. are isolated & positive (similarity principle).

can show: by making $N \gg 1$ large, can then arrange for

$$\underbrace{\int_{\mathbb{R}\times S^1} u^* d\alpha}_{\substack{\| \text{ Stokes} \\ \text{period}(\gamma_+) - \text{period}(\gamma_-)}} \text{ to be arbitrarily large} \Rightarrow \text{contra!}$$

∎

<u>We've shown</u>: $(\mathbb{T}^3, \xi_1)$ admits a ctct form α & $J \in \mathcal{J}(\alpha)$ s.t. $\mathcal{P}_2(\alpha)$ has exactly 2 orbits (one odd, even) & exactly 2 J-hol. cyls. connecting them $\Rightarrow$

$$CC^h_* = \begin{cases} \mathbb{Z}_2 & * = \text{odd} \\ \mathbb{Z}_2 & * = \text{even} \end{cases}, \quad \partial = 0.$$

∎