orientation brainteaser: Spse γ^2 is a <u>bad</u> orbit in $(M^{2n-1}, \mathcal{H})$, i.e.

$\mu_{CZ}(\gamma^2) - \mu_{CZ}(\gamma) \notin 2\mathbb{Z}$ $\Rightarrow$ for $J \in \mathcal{J}(\mathcal{H})$ on $\mathbb{R} \times M$,

$\mathcal{M}^{\$}(J) = \mathcal{M}^{\$}_{0,0}(J, J, \gamma^2; \emptyset) \circlearrowleft c$ def'd by rotating the asymp. marker $180°$

is orientation reversing. Spse $u \in \mathcal{M}^{\$,\mathrm{reg}}(J)$ has $\mathrm{ind}(u) = 1$

& $u(z) = v(z^2)$ for a curve $v : \mathbb{C} \longrightarrow \mathbb{R} \times M$ asymp. to γ.

$\overset{\mathrm{IFT}}{\Longrightarrow}$ the cpnt $\mathcal{M}_u \subseteq \mathcal{M}^{\$}(J)$ containing u consists only of $\mathbb{R}$-translations

of u, i.e. $\mathcal{M}_u \cong \mathbb{R}$. But $u(-z) = u(z)$ $\Rightarrow$ $\exists$ an automorphism

$\varphi \in \mathrm{Aut}(u)$ which rotates the marker by $180°$

$\Rightarrow$ both choices of marker at the puncture give equivalent elements of $\mathcal{M}^{\$}(J)$

$\Rightarrow$ $\mathcal{M}_u \xrightarrow{c} \mathcal{M}_u$ is Id. $\Rightarrow$ not orientation reversing!

<u>Q</u>: What went wrong?

<u>A</u>: No such <u>regular</u> curve can exist.

Spse u regular $\Rightarrow$ v also regular $\Rightarrow$ $\mathrm{ind}(v) \geq 1$.

$\mathrm{ind}(u) = (n-3)\chi(\mathbb{C}) + 2c_1^{\tau}(u) + \mu_{CZ}^{\tau}(\gamma^2) = 1$

$\Rightarrow$ $\mathrm{ind}(v) =$ " " $+ \mu_{CZ}^{\tau}(\gamma)$ is even $\Rightarrow$ ≥ 2.

$\Rightarrow$ $\exists$ a <u>2-param.</u> family of curves near u that are double covers of curves near v, impossible if $\mathrm{ind}(u) = 1$.

how to count in spite of symmetry

<u>Recall</u>: $\mathcal{M}(J) \overset{\text{loc.}}{\cong} \bar\partial_J^{-1}(0)/G$ for $G = \mathrm{Aut}(\Sigma, j_0, \Gamma \cup \Theta)$,

$\bar\partial_J : \mathcal{J} \times \mathcal{B}^{k,p,\delta} \longrightarrow \mathcal{E}^{k-1,p,\delta}$ is G-equivariant; equivalently,

$\bar\partial_J$ defines a section of the <u>orbibundle</u> $\mathcal{E}^{k-1,p,\delta}/G \longrightarrow (\mathcal{J} \times \mathcal{B}^{k,p,\delta})/G$,

whose zero set locally is $\mathcal{M}(J)$.

<u>fin.-dim. toy model</u>: $E \longrightarrow M$ = closed orbifold, E an orbibundle w/ $\mathrm{rk}\, E = \dim M$.

What is $\# \eta^{-1}(0)$ for generic $\eta \in \Gamma(E)$? (Answer should be indep of η.)

<u>ex</u>: $M = \mathbb{C}/_{z \sim -z}$, $E := M \times \mathbb{C}$, so $\Gamma(E) = \{ f : \mathbb{C} \to \mathbb{C} \mid f(-z) = f(z) \}$.

Let $f(z) = z^2$, so $f^{-1}(0) = \{[0]\}$, $\mathrm{ord}(f; 0) = 2 \implies \# f^{-1}(0) = 2$?

<u>Wrong</u>: Let $f_\varepsilon(z) := z^2 + \varepsilon$, now $f_\varepsilon^{-1}(0) = \{[\sqrt{-\varepsilon}]\}$, $df_\varepsilon(\pm\sqrt{-\varepsilon})$ is

$$[\sqrt{-\varepsilon}] = [-\sqrt{-\varepsilon}]$$

an iso. (orient. pres.) $\implies \mathrm{ord}(f_\varepsilon; \pm\sqrt{-\varepsilon}) = 1$

$$\implies \# f_\varepsilon^{-1}(0) = 1.$$

<u>thm</u>: $\forall\, \eta \in \Gamma(E)$ w/ only isolated zeroes, the number

$$\# \eta^{-1}(0) := \sum_{z \in \eta^{-1}(0)} \frac{\mathrm{ord}(\eta; z)}{\kappa_z} \in \mathbb{Q}$$

(κ_z := order of local isotropy grp. of M at z) is indep. of choice of section.

<u>pf in case</u> $M = \tilde{M}/G$ ($\tilde{M}$ a mfd, G a fin. grp.), $E = \hat{E}/G$ ($\hat{E}$ a vec. bdl)

$\Gamma(E) \ni \eta \longleftrightarrow \tilde\eta \in \Gamma(\hat{E})$ G-equivariant, then $\sum_{z \in \tilde\eta^{-1}(0)} \mathrm{ord}(\tilde\eta; z) \in \mathbb{Z}$

is indep of $\tilde\eta$. Given any $z \in \eta^{-1}(0)$, it has $\frac{|G|}{\kappa_z}$ lifts to $\tilde\eta^{-1}(0)$,

$$\implies \sum_{z \in \eta^{-1}(0)} \frac{|G|}{\kappa_z} \mathrm{ord}(\eta; z) \text{ is indep. of } \eta. \quad \square$$

rk 1: $\exists$ orbibundles for which zeroes of a section cannot be isolated.

ex: $M = \mathbb{C}/_{z \sim \bar{z}}$, $\quad E = (\mathbb{C} \times \mathbb{C})/_{(z,v) \sim (\bar{z}, -v)}$, so

$$\Gamma(E) = \{ f : \mathbb{C} \to \mathbb{C} \mid f(\bar{z}) = -f(z) \} \Rightarrow \forall f \in \Gamma(E), \ \mathbb{R} \subseteq f^{-1}(0).$$

zeroes in $\mathbb{R}$ are never regular pts.

rk 2: Even if $\eta_0, \eta_1 \in \Gamma(E)$ have isolated zeroes, sometimes $\nexists$ a htpy $\{\eta_\tau \in \Gamma(E)\}_{\tau \in [0,1]}$ for which $[0,1] \times M \to E : (\tau, z) \mapsto \eta_\tau(z)$ is $\pitchfork$ 0-section.

ex: $M = \mathbb{C}/_{z \sim -z}$, $\quad E = (\mathbb{C} \times \mathbb{C})/_{(z,v) \sim (-z,-v)}$, so $\Gamma(E) = \{ f : \mathbb{C} \to \mathbb{C} \mid f(-z) = -f(z) \}$.

Let $f_0(x+iy) = x+iy$, $\quad f_1(x+iy) = x^3 - x + iy$

$\{f_\tau\}_{\tau \in [0,1]}$ all odd fns. $\quad f_\tau(0) = 0 \ \forall \tau$

$\# f_1^{-1}(0) = 1 - \frac{1}{2} = \frac{1}{2}$

$\det df_0(0) > 0$ but $\det df_1(0) < 0 \Rightarrow \exists \ \tau_0 \in (0,1)$ s.t. $df_{\tau_0}(0)$ is singular $\Rightarrow$ the map $[0,1] \times \mathbb{C} \to \mathbb{C} : (\tau, z) \mapsto f_\tau(z)$ has a crit. pt. at $(\tau_0, 0)$.

$\Rightarrow \{ (\tau, z) \mid f_\tau(z) = 0 \}$ is not a 1-dim. cordism btwn $f_0^{-1}(0)$ & $f_1^{-1}(0)$.

solution: multivalued perturbations

Instead of $\eta \in \Gamma(E)$, consider multisections $\eta(z) \in Sym_2(E_z) := (E_z \times E_z) / (v,w) \sim (w,v)$

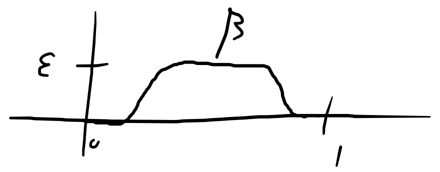
In example above, $\Gamma(E) = \{\text{odd fns.}\}$,

multisection w/ 2-branches $= g : \mathbb{C} \to Sym_2(\mathbb{C})$ s.t. $g(-z) = -g(z)$

w/ $-[(v,w)] := [(-v,-w)]$.

Defn. a htpy btwn $g_i(z) := [(f_i(z), f_i(z))]$ for $i = 0,1$ as follows:

Pick a cutoff fn.



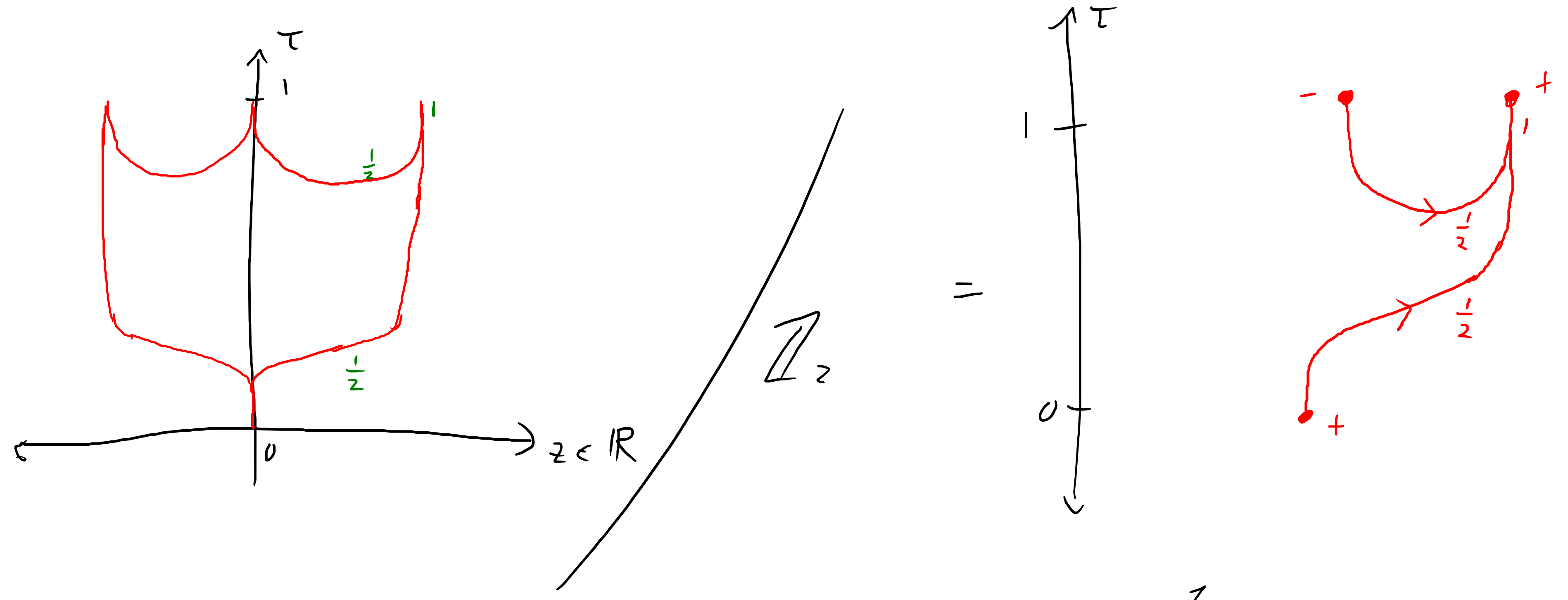
, let $g_\tau : \mathbb{C} \to Sym_2(\mathbb{C})$,

$$g_\tau(z) := [(f_\tau(z) + \beta(\tau), f_\tau(z) - \beta(\tau))].$$

$$g_\tau(-z) = [(-f_\tau(z) + \beta(\tau), -f_\tau(z) - \beta(\tau))] = -g_\tau(z)$$

$\Rightarrow g_\tau$ is a multisection of E.

Consider 0-set by attaching weight $\frac{1}{2}$ wherever only one cpt of the multisection vanishes; i.e. $f_\tau(z) = \pm\beta(\tau) \neq 0$.



$\{(\tau, z) \in [0,1] \times \mathbb{C} \mid g_\tau(z) = 0$ in at least one branch$\}$

is a cpt oriented weighted branched 1-mfd w/ bndry $= -f_0^{-1}(0) \sqcup f_1^{-1}(0)$.

thm: For any cpt oriented weight branched 1-mfd M w/ bndry, $\#\partial M = 0$ (counted w/ signs & rational weights.)