<u>yesterday</u>: ↑ sometimes requires replacing $\bar\partial_J^{-1}(0)/G$ with a weighted branched mfd $\bar\partial_J^{-1}(\{\nu_1,\dots,\nu_N\})/G$ for some G-invt. multivalued perturbation $\nu_1,\dots,\nu_N \in \Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot\Sigma, u^*T\hat W))$.

<u>standing assumption</u> (science fiction): All curves in $\mathcal{M}(J)$ are Fredholm regular (including mult. covers).

<u>Notation</u>: $\gamma = (\gamma_1,\dots,\gamma_k)$, $\quad \kappa(\gamma) := \prod_{i=1}^{k} \kappa_{\gamma_i}$ for $\kappa_{\gamma_i} := \mathrm{cov}(\gamma_i) \in \mathbb{N}$.

<u>SFT generating fn</u>

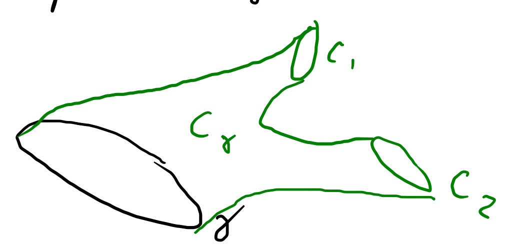
Fix $(M^{2n-1}, \xi = \ker\alpha)$, α a nondeg. ctt form, $J \in \mathcal{J}(\alpha)$ on $\mathbb{R}\times M$

+ auxiliary data:

(1) Coherent orientations $\forall$ $\mathcal{M}_{g,0}(J, A, \gamma^+, \gamma^-)$ without any bad orbits in $\gamma^\pm$

(2) For $\mathbb{Z}$-grading: assume $H_1(M)$ has no torsion (otherwise $\exists$ $\mathbb{Z}_2$-grading always).
Fix a basis of loops $C_1,\dots,C_N \subseteq M$ generating $H_1(M)$, & a triv. τ of ξ along each C_i.

(3) <u>Spanning surfaces</u> for each orbit γ: an immersed surface $C_\gamma \hookrightarrow M$ s.t. as 2-chains, $\partial C_\gamma = \sum_{i=1}^{N} m_i C_i - \gamma$.



These determine: (i) $\mu_{CZ}(\gamma) \in \mathbb{Z}$,

(ii) If u has rel. hom. class $A \in H_2(M, \bar\gamma^+ \cup \bar\gamma^-)$

$\rightsquigarrow$ absolute class $[u] := A + \sum_{z\in\Gamma^+} C_{\gamma_z} - \sum_{z\in\Gamma^-} C_{\gamma_z} \in H_2(M)$ s.t.

$$\mathrm{ind}(u) = (n-3)(2-2g-\#\Gamma) + 2\underbrace{c_1(A)}_{\|} + \sum_{z\in\Gamma^+} \mu_{CZ}(\gamma_z) - \sum_{z\in\Gamma^-} \mu_{CZ}(\gamma_z)$$

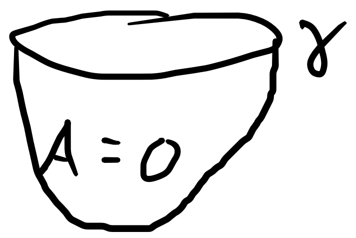
$\langle c_1(\xi), A\rangle$ for $A := [u] \in H_2(M)$.

group ring of $H_2(M)$: $R := \mathbb{Q}[H_2(M)] := \left\{ \text{finite sums } \sum_i c_i e^{A_i} \;\middle|\; c_i \in \mathbb{Q},\ A_i \in H_2(M) \right\}$

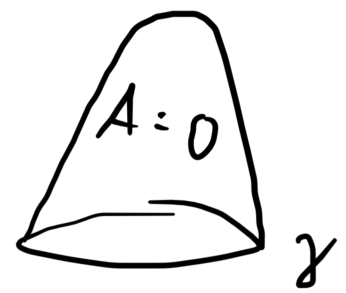
$$e^A e^B := e^{A+B}$$

grading: Each good orbit $\gamma \rightsquigarrow$ formal variables q_γ, p_γ with degrees

$$|q_\gamma| := n-3+\mu_{CZ}(\gamma), \qquad |p_\gamma| := n-3-\mu_{CZ}(\gamma)$$

$= \mathrm{ind}\Big($ 

$\Big)$

$= \mathrm{ind}\Big($ 

$\Big)$

$$|e^A| := -2c_1(A)$$

$|\hbar| := 2(n-3)$ ($\hbar$ = another variable)

$\mathcal{M}^\sigma(J) := \mathcal{M}(J) \big/$ ordering of punctures.

$\mathrm{Aut}_0(u) := \{$ bihol. maps $\varphi : \dot\Sigma \to \dot\Sigma$ (not necessarily fixing pts) s.t. $u = u \circ \varphi \}$

$\rightsquigarrow$ formal power series

$$H := \sum_{u \in \mathcal{M}_1^\sigma(J)/\mathbb{R}} \hbar^{g-1} e^A \frac{\epsilon(u)}{|\mathrm{aut}_\sigma(u)|} q^{\gamma_-} p^{\gamma_+}$$

where $\mathcal{M}_1^\sigma(J) := \{ u \in \mathcal{M}^\sigma(J) \mid \mathrm{ind}(u) = 1$ & no bad asymp. orbits $\}$

$\gamma_- = (\gamma_1^-, \ldots, \gamma_{k_-}^-)$, $\gamma_+ = (\gamma_1^+, \ldots, \gamma_{k_+}^+)$ are the neg./pos. asymp. orbits of u, w/ order chosen arbitrarily, & asymp. markers,

$$\epsilon(u) := \begin{cases} +1 & \text{if the resulting coherent orientation at } \mathbb{R}\text{-translation of } u \text{ matches the canonical one} \\ -1 & \text{otherwise} \end{cases}$$

g = genus of u, $A = [u] \in H_2(M)$,

for $\gamma = (\gamma_1, \ldots, \gamma_k)$, defn $q^\gamma := q_{\gamma_1} \cdots q_{\gamma_k}$, sim. p^γ.

Defn mult. of these generators to be supercommutative:

$$[q_\gamma, q_{\gamma'}] := q_\gamma q_{\gamma'} - (-1)^{|q_\gamma| \cdot |q_{\gamma'}|} q_{\gamma'} q_\gamma, \quad \text{sim. } p_\gamma, p_{\gamma'} \ldots$$

$\Rightarrow$ each monomial is indep. of order of orbits, also indep. of asymp. markers.

$$H = \sum_{u \in \mathcal{M}_1^\circ(\gamma)/\mathbb{R}} \frac{\epsilon(u)}{|\mathrm{aut}_\sigma(u)|} \hbar^{g-1} e^A q^{\gamma_-} p^{\gamma_+}$$

has degree $|H| = (g-1)\cdot|\hbar| + |e^A| + \sum_{z\in\Gamma^-} |q_{\gamma_z}| + \sum_{z\in\Gamma^+} |p_{\gamma_z}|$

$$= 2(n-3)(g-1) + 2c_1(A) + \sum_{z\in\Gamma^+}\left[(n-3) - \mu_{CZ}(\gamma_z)\right] + \sum_{z\in\Gamma^-}\left[(n-3) - \mu_{CZ}(\gamma_z)\right]$$

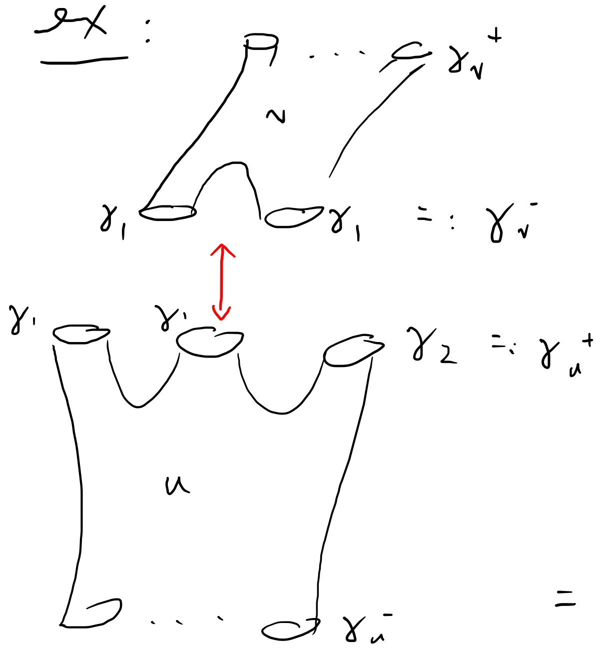
$$-\left[(n-3)(2-2g-\#\Gamma) + 2c_1(A) + \sum_{z\in\Gamma^+}\mu_{CZ}(\gamma_z) - \sum_{z\in\Gamma^-}\mu_{CZ}(\gamma_z)\right] = -1.$$

<u>thm</u>: Let $\mathcal{A}$:= graded unital algebra over $R = \mathbb{Q}[\Lambda]$ generated by $\{q_\gamma \mid \gamma \text{ good}\}$.
Then setting $p_\gamma := \kappa_\gamma \hbar \frac{\partial}{\partial q_\gamma}$ identifies H with a differential operator
$D_{SFT} := H : \mathcal{A}[[\hbar]] \to \mathcal{A}[[\hbar]]$ of degree -1 s.t. $D_{SFT}^2 = 0$.

Equivalently: setting $[p_\gamma, q_{\gamma'}] := \begin{cases} 0 & \text{if } \gamma \neq \gamma' \\ \kappa_\gamma \hbar & \text{if } \gamma = \gamma' \end{cases}$ makes $H^2 = 0$.

$$H^2 = \sum_{u\in\mathcal{M}_1^\circ(J)} \sum_{v\in\mathcal{M}_1^\circ(J)} \hbar^{g_u+g_v-2} e^{A_u+A_v} \underbrace{q^{\gamma_u^-} p^{\gamma_u^+} q^{\gamma_v^-} p^{\gamma_v^+}}_{=\,?} \cdot \frac{\epsilon(u)\,\epsilon(v)}{|\mathrm{Aut}_\sigma(u)|\cdot|\mathrm{Aut}_\sigma(v)|}$$

ex:



$$p^{\gamma_u^+} q^{\gamma_v^-} = p_1 p_1 p_2 q_1 q_1 = p_1 p_1 q_1 p_2 q_1$$

(abbreviate: $p_{\gamma_1} =: p_1$, $q_{\gamma_2} =: q_2$ etc.)

$$= p_1 (q_1 p_1 + \kappa_1 \hbar) p_2 q_1$$

$$= (q_1 p_1 + \kappa_1 \hbar) p_1 p_2 q_1 + \kappa_1 \hbar p_1 p_2 q_1$$

assume also p_1, p_2, q_1, q_2 all even generators

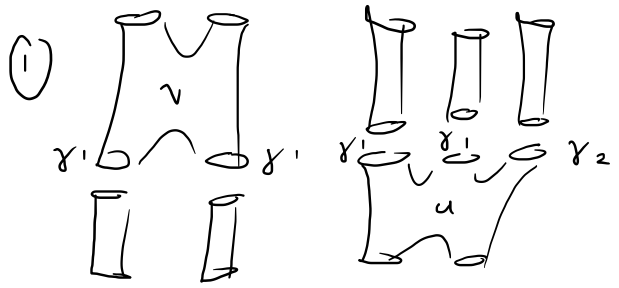
$$= (q_1 p_1 + \kappa_1 \hbar)(q_1 p_1 + \kappa_1 \hbar) p_2 + \dots$$

$$= \dots =$$

$$= q_1^2 p_1^2 p_2 + 4\kappa_1 \hbar q_1 p_1 p_2 + 2\kappa_1^2 \hbar^2 p_2$$

How many ways can u & v be glued (w/ additional triv. cyls.) to form a 2-level hol. building?

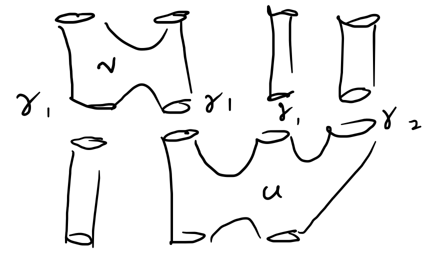
①



$$q^{\gamma_u^-} \underbrace{q_1 q_1 p_1 p_1 p_2} p^{\gamma_v^+} \hbar^{g_u+g_v-2}$$

"genus $g_u + g_v - 1$"
since ∃ 2 conn. cpts

②

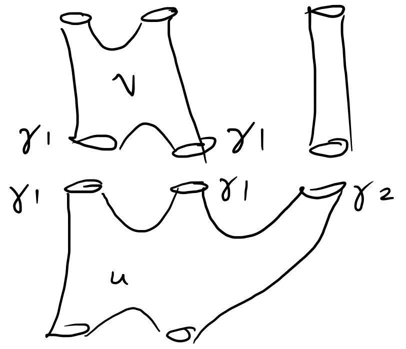


$$q^{\gamma_u^-} \underbrace{q_1 p_1 p_2} p^{\gamma_v^+} \hbar^{g_u+g_v-1}$$

genus $g_u + g_v$

Since v & u each have 2 ends at γ_1 & γ_1 has mult. $= \kappa_1$, ∃ $4\kappa_1$ distinct buildings to construct this way.

③



$$q^{\gamma_u^-} p_2 p^{\gamma_v^+} \hbar^{g_u+g_v}$$

genus
$g_u + g_v + 1$

∃ $2\kappa_1^2$ distinct buildings of this form

<u>lemma</u> ($\Rightarrow H^2 = 0$): ∀ monomial $c\, q^{\gamma^-} p^{\gamma^+} \hbar^{g-1}$ appearing in H^2, $c \in \mathbb{Q}$ is the algebraic count of 2-level hol. buildings w/ levels of index 1 & asymp. orbits γ^+, γ^-, arithmetic genus g, i.e. this is $\partial\left(\overline{\mathcal{M}_2^\circ(J)}/\mathbb{R}\right)$ for $\mathcal{M}_2^\circ(J) := \{$index 2 curves w/o bad orbits$\}$.