SFT with coefficients in $R := \mathbb{Q}[H_2(M)/G] := \left\{ \text{fin. sums } \sum_i c_i e^{A_i} \;\middle|\; \begin{array}{l} c_i \in \mathbb{Q} \\ A_i \in H_2(M)/G \end{array} \right\}$

for a subgroup $G \subseteq H_2(M)$; usually $\begin{cases} G = \{0\} \rightsquigarrow \text{fully twisted coeffs. } R = \mathbb{Q}[H_2] \\ G = H_2(M) \rightsquigarrow \text{untwisted coeffs } R = \mathbb{Q} \end{cases}$

$\mathcal{A} :=$ graded supercommutative unital algebra over R generated by $\left\{ q_\gamma \;\middle|\; \begin{array}{l} \gamma \text{ a good Reeb} \\ \text{orbit in } (M, \alpha) \end{array} \right\}$

$$H := \sum_{\underset{\text{tuples}}{\underbrace{\gamma^-, \gamma^+}}} \sum_{\substack{A \in H_2(M)/G \\ g \geq 0}} \hbar^{g-1} e^A \left(\# \underbrace{\mathcal{M}^{\sigma}_{g,0}(J, A, \gamma^+, \gamma^-)}_{\substack{\text{index 1 curves} \\ \text{modulo ordering of} \\ \text{pts homologous to any} \\ \text{class representing } A}} / \mathbb{R} \right) q^{\gamma^-} p^{\gamma^+}$$

$\in \mathbb{Q}$

<u>recall</u>: In $\mathbb{R} \times M$, $E(u)$ is bdd above in terms of γ^+, so indep. of A

$\Rightarrow$ count always finite (if $\pitchfork$ satisfied)

$$[p_\gamma, q_{\gamma'}] := p_\gamma q_\gamma - (-1)^{|p_\gamma| \cdot |q_{\gamma'}|} q_\gamma p_\gamma := \begin{cases} 0 & \text{if } \gamma \neq \gamma' \\ \kappa_\gamma \hbar & \text{if } \gamma = \gamma' \end{cases}$$

κ_γ ;; $\operatorname{cov}(\gamma) \in \mathbb{N}$

($\Leftarrow$ can represent this operator algebra by setting $p_\gamma := \kappa_\gamma \hbar \dfrac{\partial}{\partial q_\gamma}$ acting on fns of $\{q_\gamma\}$.)

$H^2 = 0 \rightsquigarrow$ can realize H as an operator $D_{SFT} : \mathcal{A}[[\hbar]] \longrightarrow \mathcal{A}[[\hbar]]$

s.t. $D_{SFT}^2 = 0 \rightsquigarrow$ homology $H_*^{SFT}(M, \xi; R)$.

<u>thm</u> (modulo $\pitchfork$): $H_*^{SFT}(M, \xi; R)$ depends only on the ctct mfd (M, ξ), (up to natural isos), not on ctct form α, $J \in \mathcal{J}(\alpha)$, coherent orientation, capping chains...

rk: D_{SFT} is not a derivation $\Rightarrow$ $H_*^{SFT}(M, \xi; R)$ is not an algebra, only an $R[[\hbar]]$-module.

variation: $[\ ,\ ]$ is a super-Lie bracket; $[A,B] = -(-1)^{|A|\cdot|B|}[B,A]$

also satisfies graded Jacobi id:

(J) $[F,[G,K]] + (-1)^{|F|\cdot(|G|+|K|)}[G,[K,F]] + (-1)^{|K|\cdot(|F|+|G|)}[K,[F,G]] = 0$

$|H| = -1$, so $H^2 = 0 \Leftarrow\Rightarrow [H,H] = 2H^2 = 0$.

Spse $V = R[[\hbar]]$-module & $L :=$ a representation of the super-Lie algebra,

i.e. $L : \{\text{fns of } (p_\gamma, q_\gamma, \hbar)\} \xrightarrow{\text{linear}} \{R[[\hbar]]\text{-linear maps } V \to V\}$

$$F \longmapsto L_F : V \to V$$

s.t. $L_{[F,G]} = L_F L_G - (-1)^{|F|\cdot|G|} L_G L_F$.

Then $L_H{}^2 = \frac{1}{2} L_{[H,H]} = \frac{1}{2}[L_H, L_H] = 0 \Rightarrow (V, L_H)$ is a chain cpx.

ex 1: $V := A[[\hbar]]$, $L_{q_\gamma} F := q_\gamma F$, $L_{p_\gamma} F := \kappa_\gamma \hbar \frac{\partial}{\partial q_\gamma} F$

& $L_{AB} := L_A L_B \Rightarrow L$ is a rep. of the Lie alg., $L_H = D_{SFT}$.

ex 2 ("full SFT" as a Weyl superalgebra)

$\leftrightarrow$ "adjoint repr"

$$\mathcal{W} := \left\{ \text{formal power series} \sum_{\gamma^+,\, g \geq 0} f_{\gamma^+, g}(q)\, p^{\gamma^+} \hbar^g \;\middle|\; \begin{array}{l} f_{\gamma^+, g}(q) = \text{polynomial in} \\ q_\gamma\text{'s w/ coeffs in } \mathcal{R} \end{array} \right\}$$

tuples of good orbits

Note: $H \notin \mathcal{W}$, but $H \in \frac{1}{\hbar}\mathcal{W}$.

Defn $L_F =: D_F : \mathcal{W} \longrightarrow \mathcal{W}$ by $D_F G := [F, G]$.

Jacobi $\Rightarrow$ $D_{[F,G]} = D_F D_G - (-1)^{|F| \cdot |G|} D_G D_F$.

observation: $\forall\ F, G \in \mathcal{W}$, $[F, G] = O(\hbar)$, i.e. $\in \hbar \mathcal{W}$

$\Rightarrow$ $D_H : \mathcal{W} \longrightarrow \mathcal{W}$ is well-def'd & $H^2 = 0 \Rightarrow D_H^2 = 0$.

$\rightsquigarrow$ $H_*^{\text{Weyl}}(M, \xi; \mathcal{R}) := H_*(\mathcal{W}, D_H)$

EX: $D_H(FG) = (D_H F) G + (-1)^{|F|} F D_H G$

$D_H [F, G] = [D_H F, G] + (-1)^{|F|} [F, D_H G]$

$\Rightarrow$ the product & bracket both descend to $H_*^{\text{Weyl}}(M, \xi; \mathcal{R})$.

semiclassical approximation ("rational SFT")

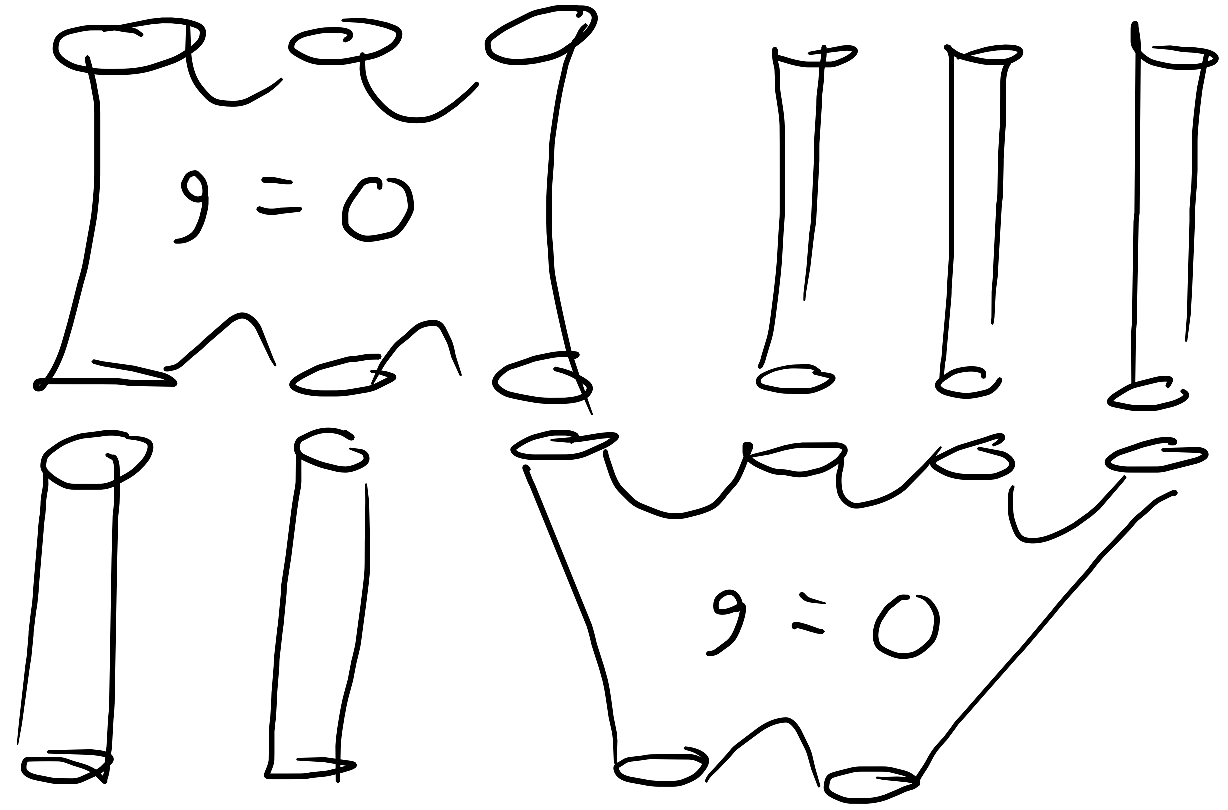
$\mathcal{P} := W / \hbar W$, same gens. p_γ, q_γ but without $\hbar$ & supercommutative

graded Poisson bracket $\{F, G\} := \sum_{\gamma \text{ good}} \kappa_\gamma \left(\frac{\partial F}{\partial p_\gamma} \frac{\partial G}{\partial q_\gamma} - (-1)^{|F| \cdot |G|} \frac{\partial G}{\partial p_\gamma} \frac{\partial F}{\partial q_\gamma} \right)$

Let $H := \frac{1}{\hbar} \sum_{g=0}^{\infty} H_g \hbar^g$, so H_g counts genus g curve.

lemma: $\hbar [H, H] = \{H_0, H_0\} + O(\hbar) \Rightarrow \{H_0, H_0\} = 0$.

Counts buildings of arithmetic $g=0$.



Can defn. $d_H : \mathcal{P} \to \mathcal{P}$ by $d_H f := \{H_0, f\}$, so Jacobi $\Rightarrow d_H^2 = 0$

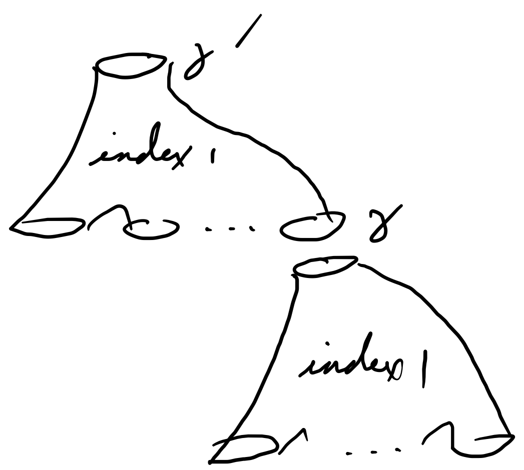
$\leadsto H_*^{RSFT}(M, \xi; \mathcal{R}) :=$

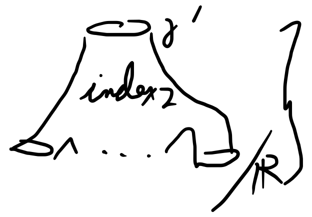
$H_*(\mathcal{P}, d_H)$, is also a graded Poisson alg.

full cont homology: Write $H_0 = \sum_\gamma h_\gamma(q)\, p_\gamma + O(p^2)$, then

$$\{H_0, H_0\} = \sum_{\gamma,\gamma'} 2\kappa_\gamma \big(h_\gamma(q) + O(p)\big)\left(\frac{\partial h_{\gamma'}(q)}{\partial q_\gamma}\, p_{\gamma'} + O(p^2)\right) = 0$$

$\Rightarrow \forall\, \gamma',\ \sum_\gamma \kappa_\gamma h_\gamma(q) \frac{\partial h_{\gamma'}}{\partial q_\gamma} = 0$ counts



$= \partial \Big\{$  $/\mathbb{R} \Big\}$ w/ any # of negative ends.

Defn $\partial_{CH} : \mathcal{A} \longrightarrow \mathcal{A}$ by $\partial_{CH} f := \{H_0, f\}\big|_{p=0}$, i.e.

$$\partial_{CH}\, q_\gamma = \{h_\gamma(q)\, p_\gamma,\ q_\gamma\} = \kappa_\gamma \frac{\partial(h_\gamma p_\gamma)}{\partial p_\gamma} = \kappa_\gamma h_\gamma$$

$= \sum_{\gamma^-, A} \kappa_\gamma e^A \,\#\Big($  $/\mathbb{R}\Big)\, q^{\gamma^-}$.

tuples of good orbits

$\rightsquigarrow H_*^{CH}(M, \xi; \mathcal{R}) := H_*(\mathcal{A}, \partial_{CH})$

EX: ∂_{CH} is a derivation on $\mathcal{A}$, $\Rightarrow$ product descends to homology.

defn: (M, ξ) is algebraically overtwisted if $H_*^{CH}(M, \xi; \mathcal{R}) = \{0\}\ \forall\, \mathcal{R}$.
(note: $\Leftrightarrow\ 1 = 0$)

thm 1: (M, ξ) overtwisted $\Rightarrow$ alg. O.T. (due to Eliashberg in dim 3)

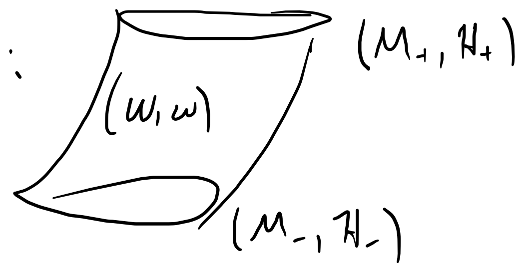
thm 2: alg O.T. $\Leftrightarrow H_*^{Weyl}(M, \xi; \mathcal{R})\ \forall\, \mathcal{R} \Leftrightarrow H_*^{RSFT}(M, \xi; \mathcal{R})\ \forall\, \mathcal{R}$. (Bourgeois–Niederkrüger)

Q 1: Alg O.T. $\Rightarrow$ O.T.?

Q 2: $\exists\ (M, \xi)$ s.t. $H_*^{CH}(M, \xi; \mathcal{R}) = 0$ for some $\mathcal{R}$ but all?

OPEN

Cobordisms :



assume SHS's $\mathcal{H}_\pm = (\omega_\pm, \alpha_\pm)$
ctct forms

$\Rightarrow$ Reeb vec. flds for $\mathcal{H}_\pm$ are same as for $\alpha_\pm$.

Assume SFT gen. fns. $H_\pm$ can be def'd by counting index 1 in $(\mathbb{R}\times M_\pm, J_\pm)$ for $J_\pm \in \mathcal{J}(\mathcal{H}_\pm)$, also $\pitchfork$ always satisfied.

$\mathcal{M}_0^\circ(J) := \{$ stable index 0 curves in $\hat{W}$ asymp. to good orbits $\}$ / ordering of pts.

"stable": excluding constant curves of $g=0$ or 1

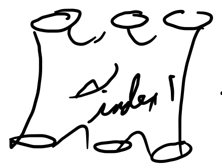
$\rightsquigarrow$ formal power series

$$F := \sum_{u \in \mathcal{M}_0^\circ(J)} \frac{\epsilon(u)}{|\mathrm{Aut}^\circ(u)|} \hbar^{g-1} e^A q^{\gamma^-} p^{\gamma^+}$$

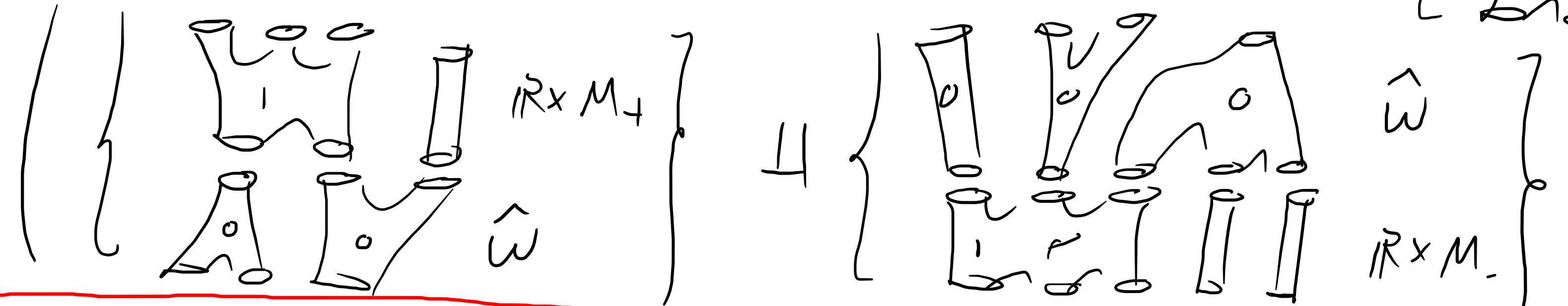
where $\epsilon(u) = \pm 1$ (coh. orient.)
g = genus of u, $\gamma^\pm$ its asymp. orbits, $A \in H_2(W)$ w.r.t. choices of capping chains

SFT cptness $\Rightarrow$ coeff. in front of $\hbar^{g-1} q^{\gamma^-} p^{\gamma^+}$ in F belongs to the Novikov ring $\left\{ \sum_{i=1}^\infty c_i e^{A_i} \,\middle|\, c_i \in \mathbb{Q},\ A_i \in H_2(W),\ \lim_{i\to\infty} \langle [\omega], A_i \rangle = \infty \right\}$.

case $M_\pm = \emptyset$: (W,ω) closed, F = "Gromov-Witten potential"

SFT master eqn: $0 = \# \partial \overline{\mathcal{M}_1^\circ(J)} = \# \partial \{$  in $\hat{W}$ $\}$

$= \# \Big(\{$



$\} \sqcup \{ \ldots \} \Big)$

$$\Rightarrow \boxed{H_- \exp(F)\big|_{p_-=0} - \exp(F) H_+\big|_{q_+=0} = 0}$$

$\exp(F) = 1 + F + \frac{1}{2}F^2 + \ldots$ counts possibly disconnected index 0 curves in $\hat{W}$

empty curve — connected curves — 2 cpts