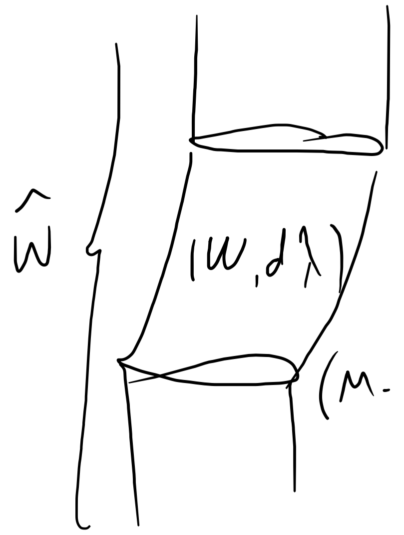


Exact sympl. cobordism ("Liouville cob."):

λ = primitive of ω on W s.t. $\lambda|_{M_\pm} = \alpha_\pm$ ctct.

$J \in \mathcal{J}(W, d\lambda)$ s.t. extends to ends of $\hat W$ as $J_\pm \in \mathcal{J}(\alpha_\pm)$.

$$\rightsquigarrow H_\pm = \sum_{\gamma^+, \gamma^-, g, A} \hbar^{g-1} e^A \left(\#\mathcal{M}^\sigma_{g,0}(J_\pm, A, \gamma^+, \gamma^-)/\mathbb{R}\right) q^{\gamma^-} p^{\gamma^+}$$

counts curves in $(\mathbb{R}\times M_\pm, J_\pm)$ of index 1.

F := similar defn for counting index 0 curves in $(\hat W, J)$

$\#\partial\{$index 1 curves in $\hat W\} = 0 \Rightarrow H_- \exp(F)|_{p_-=0} - \exp(F) H_+|_{q_+=0} = 0.$

Rk: In defn. of F, can take $A \in H_2(W)/G$ for any subgroup G since $\langle [d\lambda], A\rangle = 0 \quad \forall A \in H_2(W)$.

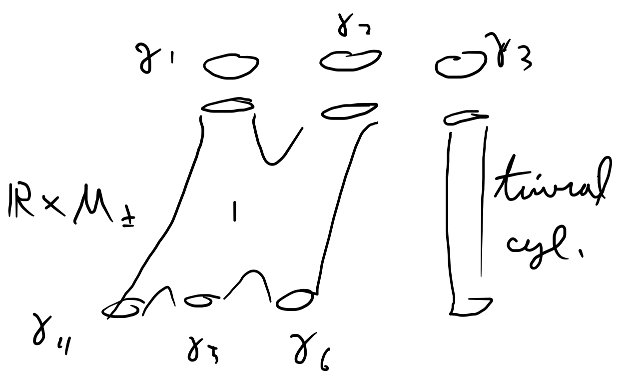
In fact, $d\lambda$ exact $\Rightarrow E(u)$ is bdd above in terms of pos. asymp. orbits $\Rightarrow$ coeffs in F for each $\hbar^{g-1} q^{\gamma^-} p^{\gamma^+}$ are in $R := \mathbb{Q}[H_2(W)/G]$, not its Novikov completion.

Also, $F = O(p)$, i.e. all terms contain at least one p_γ.

For now: take $G = H_2(W)$, i.e. $R = \mathbb{Q}$ ("untwisted coeffs"); sim. in $H_\pm$.

Recall: $H_*^{SFT}(M, \xi_\pm) := H_*(\mathcal{A}^\pm[[\hbar]], D^\pm_{SFT})$, $D^\pm_{SFT} := H_\pm$ via substitution

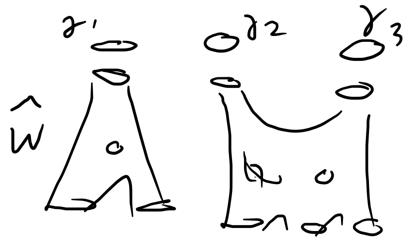
$\mathcal{A}^\pm$ gen. by q_γ for $\gamma \in M_\pm$; $p_\gamma := \kappa_\gamma \hbar \frac{\partial}{\partial q_\gamma}$.



Let $\Phi : \mathcal{A}^+[[\hbar]] \longrightarrow \mathcal{A}^-[[\hbar]]$

$f \longmapsto \exp(F) f|_{q_+=0}$

where again $p_\gamma = \kappa_\gamma \hbar \frac{\partial}{\partial q_\gamma}$ for each γ in M_+.



Master eqn $\Rightarrow \Phi$ is a chain map $\Rightarrow$ induces a $\mathbb{Q}[[\hbar]]$-linear map $H_*^{SFT}(M_+, \xi_+) \longrightarrow H_*^{SFT}(M_-, \xi_-)$.

application 1 : invariance

$W := [0,1] \times M$, so $(M_\pm, \xi_\pm) = (M, \xi)$ w/ different choices of ctct form $\alpha_\pm$ at $M_+ := \{1\} \times M$ & $M_- := \{0\} \times M$, $J_\pm \in \mathcal{J}(\alpha_\pm)$, $J \in \mathcal{J}([0,1] \times M, d\lambda)$. Sim. arg. as in CCH $\Rightarrow$ chain map Φ has a chain htpy inverse $\Rightarrow$ up to natural isos., $H_*^{SFT}(M, \xi)$ is indep. of choices of α & J.

application 2 : cobordism obstructions

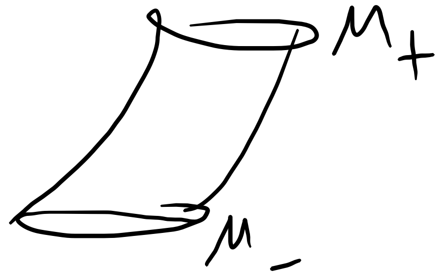
Recall : $\nexists$ topological obstruction to an almost cpx cobordism btwn any 2 given ctct mfds.

defn : $k \geq 0$ integer, (M, ξ) has algebraic k-torsion if $[\hbar^k] = 0 \in H_*^{SFT}(M, \xi)$.

$AT(M, \xi) := \sup \{ k \geq 0 \mid [\hbar^{k-1}] \neq 0 \in H_*^{SFT}(M, \xi) \} \in \mathbb{N} \cup \{0, \infty\}$.

rk (follows by a variation on Bourgeois–Niederkrüger):

$AT(M, \xi) = 0 \iff [1] = 0 \in H_*^{SFT}(M, \xi) \iff H_*^{SFT}(M, \xi) = \{0\} \iff (M, \xi)$ is alg. overtwisted.

thm 1: If $\exists$ exact cob. 

, then $AT(M_-) \leq AT(M_+)$.

In particular, $H_*^{SFT}(\emptyset) = \mathbb{Q}[[\hbar]] \Rightarrow AT(\emptyset) = \infty$,

cor: If $AT(M) < \infty$, then M has no exact symp. fillings. $\boxtimes$

pf of thm 1: If $AT(M_+) = k$, then $[\hbar^k] = 0 \in H_*^{SFT}(M_+)$,

the chain map $\Phi : H_*^{SFT}(M_+) \longrightarrow H_*^{SFT}(M_-)$ sends $[\hbar^k] \longmapsto [\hbar^k]$, $0 \longmapsto 0$

$\Rightarrow [\hbar^k] = 0 \in H_*^{SFT}(M_-)$. $\boxtimes$

defn: (W, ω) is a <u>weak</u> symp. filling of $(M, \xi = \ker \alpha)$ if $M = \partial W$

& $\exists\ J \in \mathcal{J}_\tau(W, \omega)$ s.t. $TM \cap J(TM) = \xi$ & $d\alpha|_\xi$ tames $J|_\xi$.

EX: For $\dim M = 3$, weak filling $\Longleftrightarrow \omega|_\xi > 0$.

ex: $(\mathbb{T}^3, \xi_k)$ is isotopic to $(\mathbb{T}^3$, small perturbation of $\ker d\rho = \mathrm{Span}\{\partial_\phi, \partial_\theta\})$

$(k \in \mathbb{N})$ $\quad \mathbb{T}^3 \ni (\rho, \phi, \theta) \quad \Rightarrow (\mathbb{T}^3, \xi_k)$ is weakly filled by

$$\Big(\mathbb{D}^2 \times \underset{\substack{\cup \\ (\phi,\theta)}}{\mathbb{T}^2},\ (\text{area form})_{\mathbb{D}^2} \oplus (\text{area form})_{\mathbb{T}^2}\Big).$$

recall: (W, ω) <u>strong</u> filling of (M, ξ) means near ∂W, $\omega = d\lambda$ s.t.

$\lambda|_{TM} = \alpha$: cont form for ξ.

Lemma (Niederkrüger - W. / Cieliebak - Volkov): Up to deformation, a weak filling (W, ω) of $(M, \xi = \ker \alpha)$ can always be assumed to have stable boundary, inducing a SHS $\mathcal{H}_c := (\Omega + c\, d\alpha, \alpha)$ on M $\left(\begin{array}{c}\Omega = \text{closed 2-form,} \\ c \gg 0\end{array}\right)$.

rk: $\mathcal{J}(\mathcal{H}_c) = \mathcal{J}\left(\frac{1}{c}\Omega + d\alpha, \alpha\right) \Rightarrow$ for $c \gg 0$, can always take $J \in \mathcal{J}(\mathcal{H}_c)$ C^∞-close to something in $\mathcal{J}(\alpha)$.

thm 2: Spse $\beta \in H^2_{dR}(M)$, set $G := \{A \in H_2(M) \mid \langle \beta, A \rangle = 0\}$, $\mathcal{R} := \mathbb{Q}[H_2(M)/G]$.

If $[\hbar^k] = 0 \in H_*^{SFT}(M, \xi; \mathcal{R})$, then (M, ξ) has no weak filling (W, ω) w/ $[\omega|_{TM}] = \beta$.

$\Rightarrow$ (i) $\nexists$ strong fillings if (M, ξ) has alg. k-torsion (case $\beta = 0$, $\mathcal{R} = \mathbb{Q}$).

$\Rightarrow$ (ii) $\nexists$ weak fillings if $[\hbar^k] = 0 \in H_*^{SFT}(M, \xi; \mathbb{Q}[H_2(M)])$

("fully twisted alg. torsion")

cor: Alg. overtwisted $\Rightarrow$ $\nexists$ weak fillings.

pf: Given a weak filling (W, ω), WLOG stable at ∂W, can choose coeffs. $R_0 := \mathbb{Q}[H_2(W)/\ker[\omega]]$ & use generating fn F to defn a chain map $\Phi : H_*^{SFT}(M, \xi; \mathcal{R}) \longrightarrow \overline{R}_0[[\hbar]]$ where $\overline{R}_0 :=$ Novikov completion of R_0.

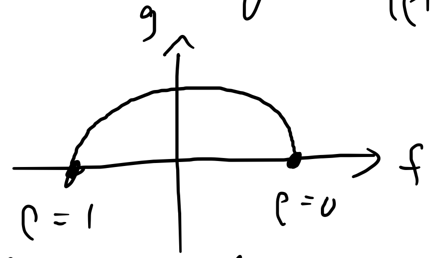
Then $[\hbar^k] = 0 \Rightarrow \Phi[\hbar^k] = \hbar^k = 0 \in \overline{R}_0[[\hbar]]$ contradiction. ∎

important detail: $M \hookrightarrow W$ induces a map $H_2(M) \to H_2(W)$ that descends to $H_2(M)/\ker[\omega|_{TM}] \longrightarrow H_2(W)/\ker[\omega]$.

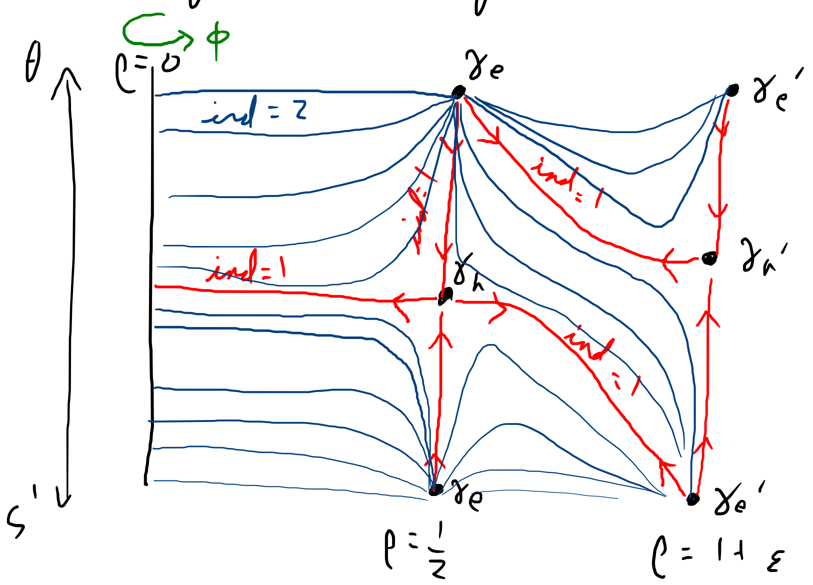
computations:

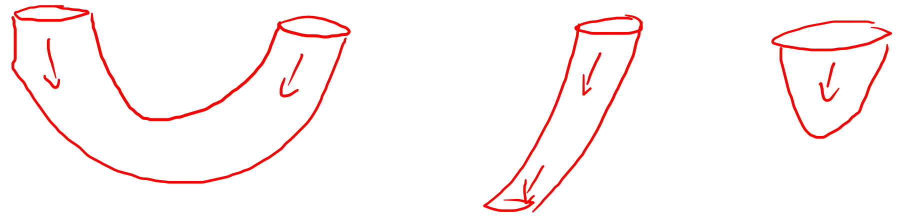
thm: (M^3,ξ) overtwisted $\Rightarrow$ $[1] = 0 \in H_*^{CH}(M,\xi;\mathbb{Q}[H_2(M)])$.

Pf: O.T. $\Longleftrightarrow$ $\exists$ $(\underset{\theta}{S^1} \times \underset{(\rho,\phi)}{\mathbb{D}^2}, \xi_{OT}) \xrightarrow{\text{emb.}} (M,\xi)$, $\xi_{OT} := \ker[f(\rho)\,d\theta + g(\rho)\,d\phi]$

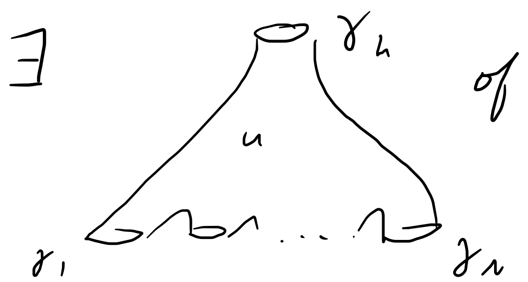
where 

By explicit construction of a ctct form α & rotationally symmetric $J \in \mathcal{J}(\alpha)$, can find a foliation of $(\mathbb{R}\times(S^1\times\mathbb{D}^2_{1+\varepsilon},\xi_{OT}),J)$ by J-hol. planes & cylinders, & this projects to a foliation of $S^1\times\mathbb{D}^2_{1+\varepsilon}$ outside a finite set of Reeb orbits:

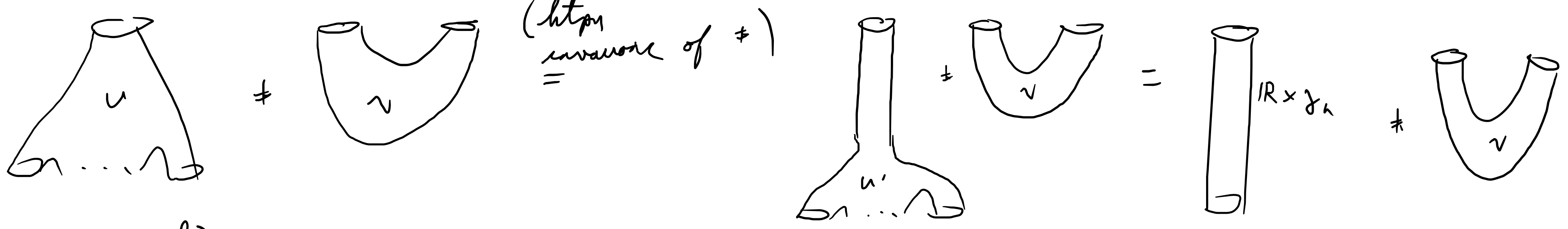




claim: $\partial_{CH} q_{\gamma_h} = 1$. ($\Rightarrow [1] = 0$.)

Pf: Suppose $\exists$  of ind = 1.

Then u intersects curves in the foliation; in particular, $\exists$ a curve v in the foliation w/ only pos. ends s.t. $u * v > 0$ (Siefring intersection #).



(last week) $=$ 0 contra, unless u is a curve in the foliation.

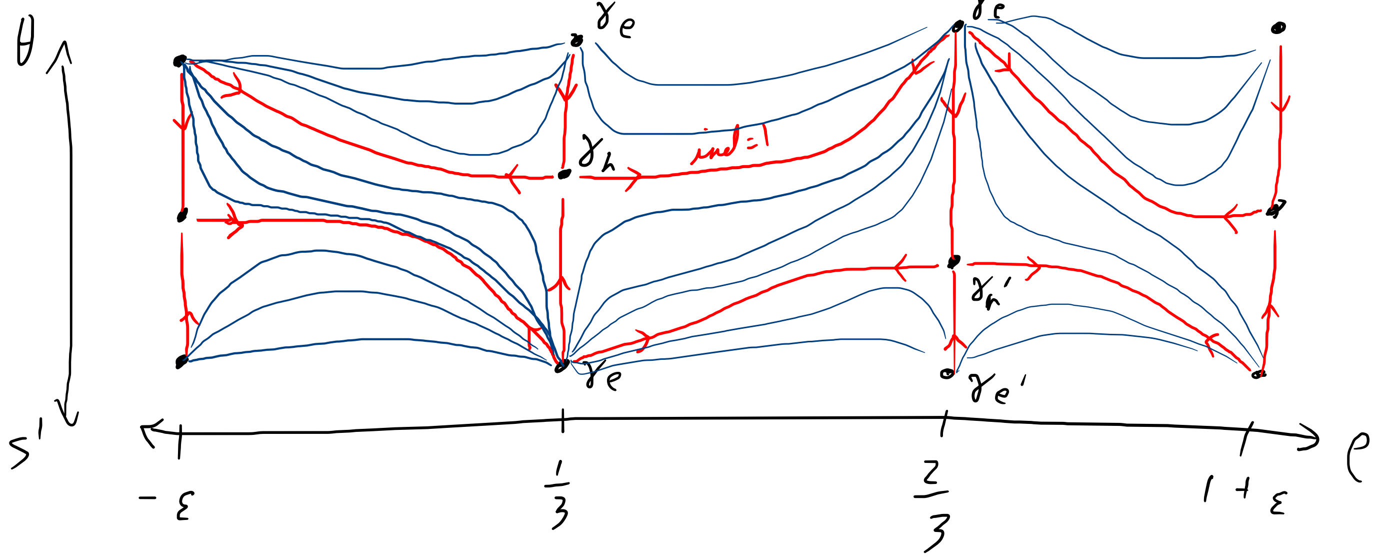
$\Rightarrow$ $\exists$ only one plane  to count, $\partial_{CH} q_{\gamma_h} = 1$. □

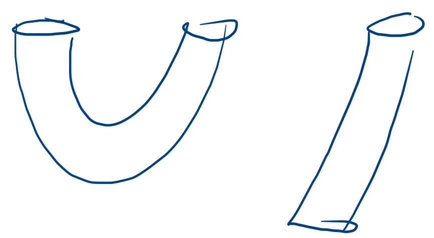
Giroux torsion: $([0,1] \times \mathbb{T}^2, \xi_{GT}) \xrightarrow{emb} (M^3, \xi)$

$\rho \in [0,1]$, $(\phi, \theta) \in \mathbb{T}^2$

$\xi_{GT} = \ker\left[\cos(2\pi\rho)\, d\theta + \sin(2\pi\rho)\, d\phi\right]$

explicit construction of $\rightsquigarrow \alpha, J$

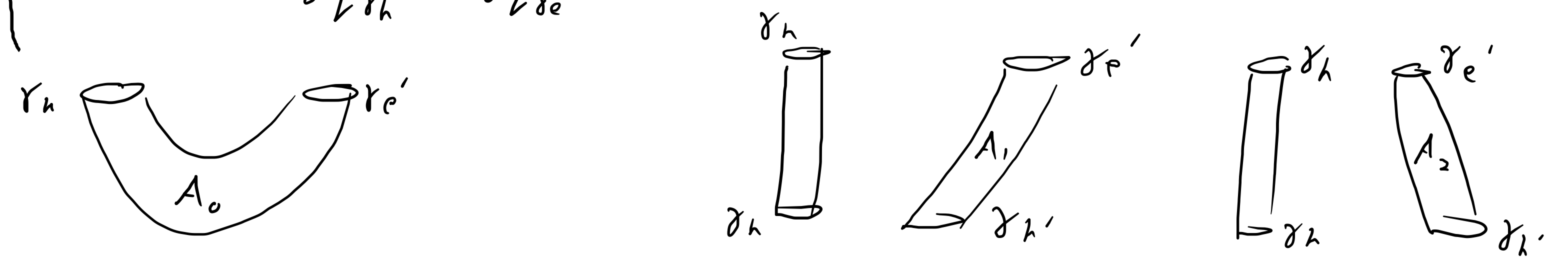




$D_{SFT} : \mathcal{A}[[\hbar]] \longrightarrow \mathcal{A}[[\hbar]]$

$$= \sum_{\text{index curves}} \hbar^{g-1} e^{A} q^{\gamma^-} p^{\gamma^+} \quad \text{where} \quad p_\gamma = \kappa_\gamma \hbar \frac{\partial}{\partial q_\gamma}$$

$$D_{SFT}(q_{\gamma_h} q_{\gamma_e'}) = \left(\hbar^{-1} e^{A_0} \hbar \frac{\partial}{\partial q_{\gamma_h}} \hbar \frac{\partial}{\partial q_{\gamma_e'}} + \hbar^{-1} e^{A_1} q_{\gamma_h'} \hbar \frac{\partial}{\partial q_{\gamma_e'}} - \hbar^{-1} e^{A_2} q_{\gamma_h'} \hbar \frac{\partial}{\partial q_{\gamma_e'}}\right) q_{\gamma_h} q_{\gamma_e'}$$



$$= \hbar e^{A_0} + (e^{A_1} - e^{A_2})\, q_{\gamma_h'} q_{\gamma_h}$$

$$\Rightarrow D_{SFT}(e^{-A_0} q_{\gamma_h} q_{\gamma_e'}) = \begin{cases} \hbar & \text{if } A_1 - A_2 = 0 \in H_2(M)/G \\ \hbar + \text{something else} & \text{otherwise.} \end{cases}$$

$\Rightarrow$ <u>thm</u>: (M, ξ) has positive Giroux torsion, then it has alg. 1-torsion.

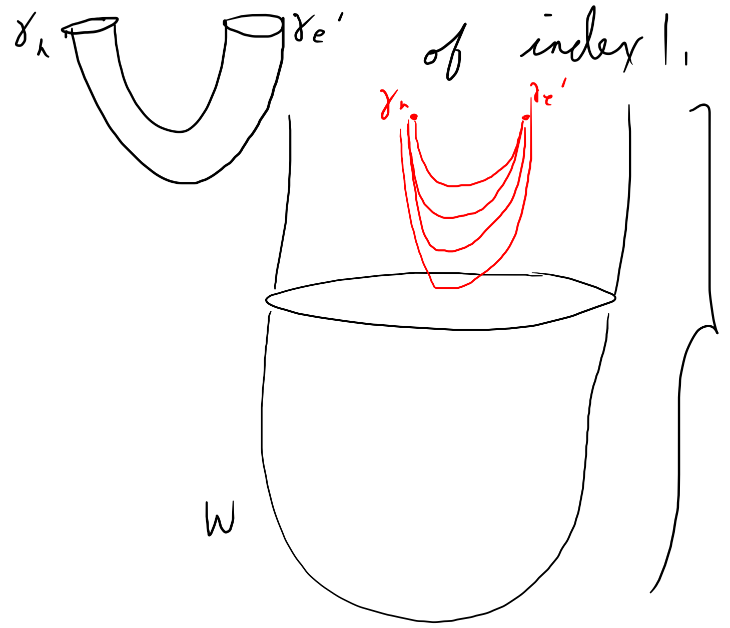
Moreover, if the torus $\{0\} \times \mathbb{T}^2 \longrightarrow M$ separates M, then $[\hbar] = 0 \in H_*^{SFT}(M, \xi; \mathbb{Q}[H_2(M)])$.

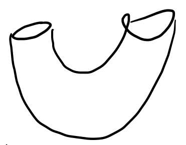
<u>ex</u>: $(\mathbb{T}^3, \xi_k)$ is strongly fillable <u>only</u> for $k = 1$, but they are all weakly fillable.

thm: If (M^3,ξ) has G.T., then not strongly fillable.

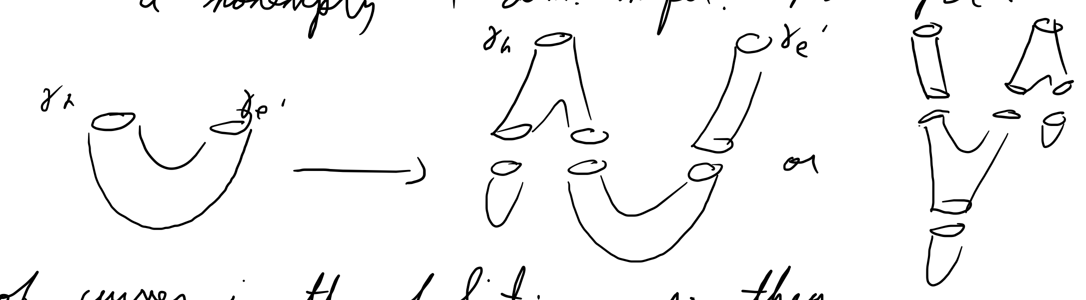
rigorous pf: Sps (W,ω) is a filling of (M,ξ), ⇝ completion $\hat{W}$ contains $[0,\infty)\times M$, choose compatible J s.t. $J_+ := J|_{[0,\infty)\times M} \in \mathcal{J}(\alpha)$ admits the foliation by hol. curves in $\mathbb{R}\times M$ on the previous page

⇒ the cyl. end of $\hat{W}$ contains a 1-parameter family of J-hol. cyls. γ_h γ_e' of index 1, all regular.

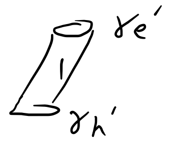
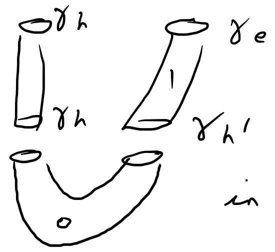


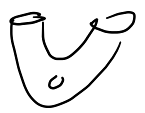
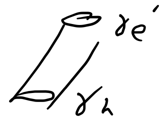
Let $\mathcal{M}_1$:= moduli space of J-hol cylinders γ_h  γ_e' in $\hat{W}$.

This is (for generic perturbation of J in $\mathring{W}$) a nonempty 1-dim. mfd. Not cpt:

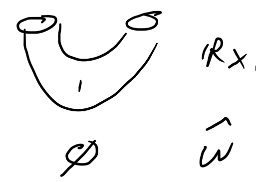


Top level can consist only of curves in the foliation, so they are either (i)  (⇒ main level is empty), or

(ii) One of 2 cylinders  (⇒  in $\mathbb{R}\times M$, in $\hat{W}$).

observation: ∃ exactly 2 buildings of type (ii) with any given index 0 curve γ_h  γ_h' in main level; using automatic ⋔, can show the 2 index 1 cyls.  have opposite signs ∀ choices of coh. orientations.

Defn $\hat{\mathcal{M}}_1 := \bar{\mathcal{M}}_1/\sim$ where buildings $u \sim v$ iff their main levels are identical.

⇒ $\hat{\mathcal{M}}_1$ is a cpt 1-mfd whose body is one pt. ( $\mathbb{R}\times M$, $\emptyset$ $\hat{W}$)

∎