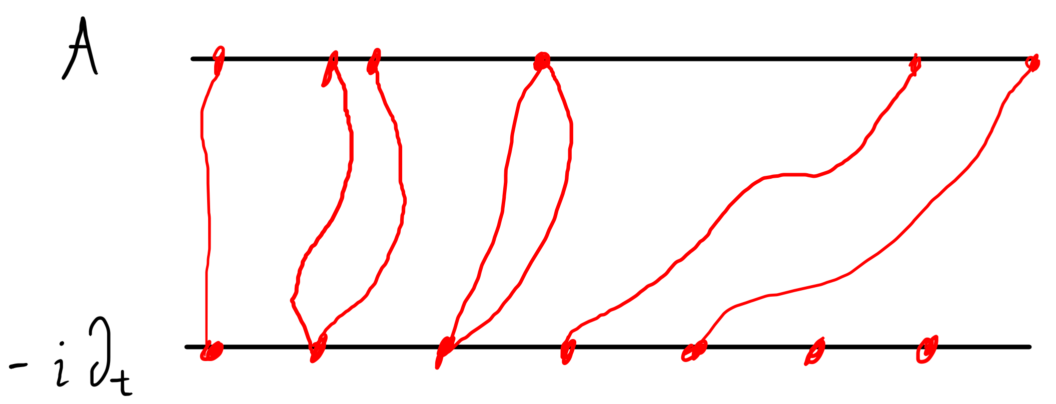


$\Rightarrow$ crit. pts. of the ctct action fnal always have infinite Morse index & Morse co-index!

<u>AGENDA</u>: (1) CZ-index + dynamics

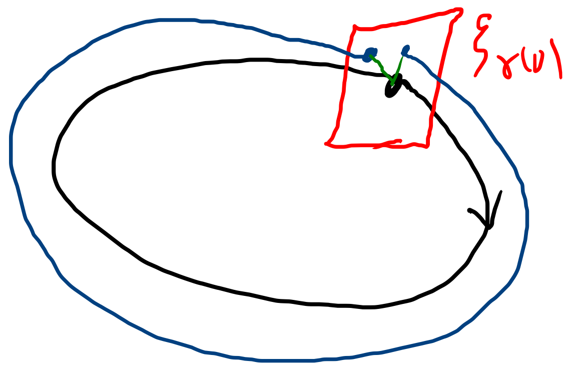
(2) Ctct sts. & orbits on $\mathbb{T}^3$

<u>dynamics</u>: $(M^{2n-1}, \xi = \ker\alpha)$ ctct mfd, Reeb vec. fld R_α

$\rightsquigarrow$ flow $\varphi^t : M \to M$ s.t. for $\gamma : \mathbb{R} \to M$ an orbit of R_α,

$\varphi^t_* : \xi_{\gamma(0)} \to \xi_{\gamma(t)}$, in fact this is sympl. iso. $(\xi_{\gamma(0)}, d\alpha|_\xi) \to (\xi_{\gamma(t)}, d\alpha|_\xi)$.

If γ has period $T > 0 \rightsquigarrow \varphi^T_* : (\xi_{\gamma(0)}, d\alpha|_\xi) \circlearrowleft$, i.e. in coords, this is a linear map $Sp(2n-2) \subseteq SL(2n-2, \mathbb{R})$.



<u>defn</u>: γ is <u>nondegenerate</u> if $1 \notin \sigma(\varphi^T_*|_{\xi_{\gamma(0)}})$.

($\Rightarrow$ $\nexists$ other periodic orbits C^∞-close to $\gamma : \mathbb{R}/T\mathbb{Z} \to M$ except reparametrizations of γ.)

<u>rk</u>: For any orbit $\gamma : \mathbb{R} \to M$, $\exists!$ sympl. connection ∇^α on $\gamma^*\xi$ s.t. $\varphi^t_* : \xi_{\gamma(0)} \to \xi_{\gamma(t)}$ is parallel transport.

nondeg. $\Leftrightarrow$ $\nexists$ parallel section (w.r.t. ∇^α) of $\gamma^*\xi$ (for $\gamma : \mathbb{R}/T\mathbb{Z} \to M$)

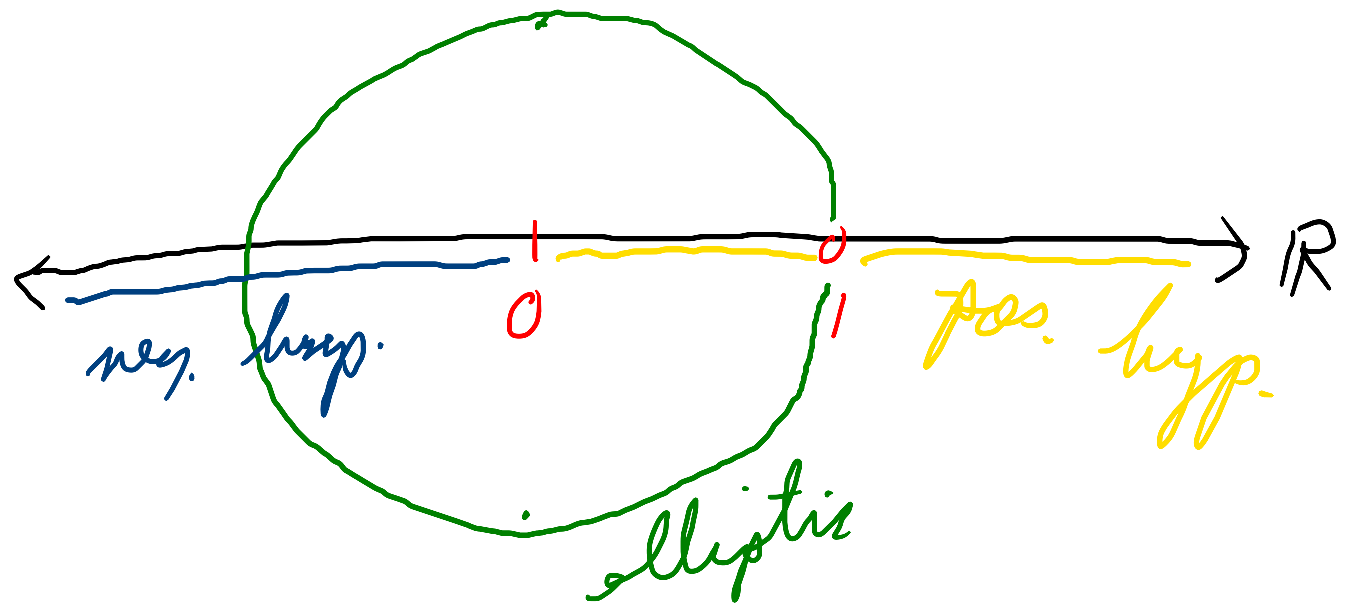
$\Leftrightarrow A_\gamma = -J\nabla^\alpha_t$ has trivial kernel.

defn: For $\dim M = 3$, $\varphi_*^T : \xi_{\gamma(0)} \circlearrowleft$ has 2 evals. $\lambda_1, \lambda_2 \in \mathbb{C}$, $\lambda_1 \lambda_2 = 1$.

γ is positive hyperbolic if $\lambda_1, \lambda_2 > 0$.

γ is neg. hyp. if $\lambda_1, \lambda_2 < 0$.

elliptic if $\lambda_1, \lambda_2 \notin \mathbb{R}$ $\left(\Leftrightarrow \; = e^{\pm i\theta} \right)$.



$\nexists$ nondeg. deformation elliptic $\longleftrightarrow$ positive hyp.

(elliptic $\longleftrightarrow$ neg. hyp. $\Rightarrow$ double cover of γ becomes degenerate)

Consider a trivial Hermitian line bndl $S^1 \times \overset{\mathbb{R}^2}{\mathbb{C}}$, $J = i \overset{\text{"}}{=} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\omega = \omega_{std}$

w/ a symp. conn. $\nabla \rightsquigarrow$ asymp. op. $A = -i \nabla_t$.

Par. transp. wrt $\nabla \rightsquigarrow$ family $\{ \Phi_t \in Sp(2) \}_{t \in [0,1]}$ s.t. $\Phi_0 = Id$ &

A degenerate iff $1 \in \sigma(\Phi_1)$.

$\Phi_t = !$ sol. to $\begin{cases} \nabla_t \Phi_t = 0 \\ \Phi_0 = Id \end{cases}$ $\Longleftarrow$ if $A = -i\partial_t - S$, this means

$(-i\partial_t - S)\Phi = 0 \Leftrightarrow \dot{\Phi} = \underset{\overset{\text{"}}{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}}{i} \underset{\mathbb{R}\text{-linear}}{S} \Phi$.

elliptic model: $A = -i\partial_t - \varepsilon$ for $\varepsilon \in \mathbb{R}$ const., $\sigma(A) = 2\pi\mathbb{Z} - \varepsilon$,

i.e. A nondeg. iff $\varepsilon \notin 2\pi\mathbb{Z}$. $\dot{\Phi} = iS\Phi = i\varepsilon\Phi \Rightarrow$

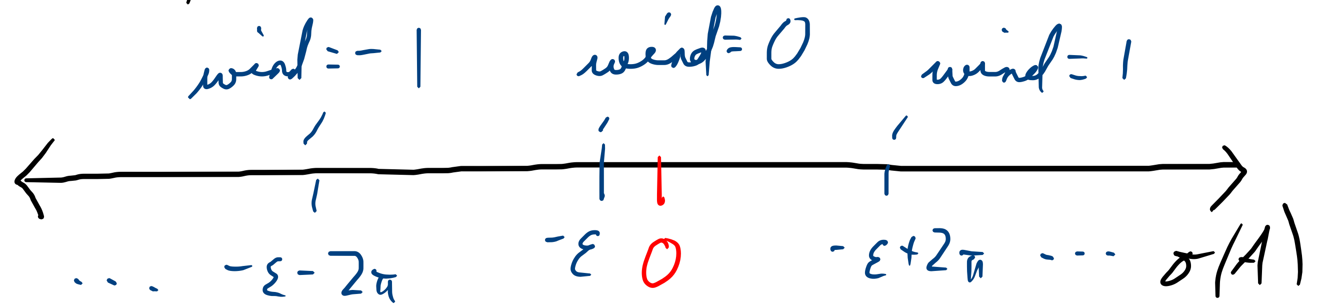
$$\Phi(t) = e^{i\varepsilon t} \in GL(2,\mathbb{R}) \quad (\text{via identification } \mathbb{C} = \mathbb{R}^2)$$

$$\parallel$$

$$\begin{pmatrix} \cos\varepsilon t & -\sin\varepsilon t \\ \sin\varepsilon t & \cos\varepsilon t \end{pmatrix}, \quad \sigma(\Phi(t)) = \{e^{i\varepsilon t}, e^{-i\varepsilon t}\}.$$

$t = 1$: $\sigma(\Phi(1)) \not\subset \mathbb{R} \Rightarrow$ elliptic.

CZ-index: $\mu_{CZ}(A) = 2\underbrace{\alpha_-(A)}_{\substack{\text{largest winding} \\ \text{for e-val.} < 0}} + p(A)$



If $\varepsilon > 0$ small

$\mu_{CZ}(A) = 1$.

$\alpha_-(A) = 0$, $\alpha_+(A) = 1$

$\Rightarrow p(A) = 1$

If $2\pi(k-1) < \varepsilon < 2\pi k$, similarly $\alpha_-(A) = k$, $\alpha_+(A) = k+1$. $\mu_{CZ}(A) = 2k+1$.

rk: $\mu_{CZ}^{\tau}(\gamma) \in \mathbb{Z}$ deps on a choice of triv. τ of $\gamma^*\xi$.

changing τ changes $\alpha_\pm(A)$ in some way $\Rightarrow$ does not change $p(A)$

$\Rightarrow \mu_{CZ}^{\tau}(\gamma) \pmod 2 \in \mathbb{Z}_2$ is indep. of τ.

thm: γ elliptic or neg. hyp. $\Leftrightarrow$ $\mu_{CZ}^{\tau}(\gamma)$ odd.

<u>pos. hyperbolic model</u>: $A := -i\partial_t - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mu_{cz}(A) = 0$ by defn.

$$\dot{\Phi} = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Phi$$

$$\sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \{1, -1\} \Rightarrow \sigma(\Phi(t)) = \{e^t, e^{-t}\} \Rightarrow \text{pos. hyperbolic.}$$

$\Rightarrow$ <u>thm</u>: γ pos. hyperbolic $\Leftrightarrow$ $\mu^{\tau}_{cz}(\gamma)$ even.

(<u>thm</u>: Let $\gamma^k :=$ k-fold cover of γ.

For nondeg. ctct forms α on M^3, to each simply covered orbit γ:

- If γ hyperbolic, then $\mu^{\tau^k}_{cz}(\gamma^k) = k\,\mu^{\tau}_{cz}(\gamma)$.
- If γ elliptic, then $\exists\ \theta \in \mathbb{R} \setminus \mathbb{Q}$ s.t.

$$\mu^{\tau^k}_{cz}(\gamma^k) = 2\lfloor k\theta \rfloor + 1. \quad)$$

ctct strs on $\mathbb{T}^3$

$\mathbb{T}^3 = S^1 \times S^1 \times S^1 \ni (\rho, \phi, \theta)$, $\alpha_k := \underbrace{\cos(2\pi k\rho)}_{f(\rho)}\, d\theta + \underbrace{\sin(2\pi k \rho)}_{g(\rho)}\, d\phi$

$\xi_k := \ker \alpha_k \subseteq T\mathbb{T}^3$.

For $c \in \mathbb{R}$, $\lambda^c := \alpha_k + c\, d\rho$, then

$$\lambda^c \wedge d\lambda^c = (f\, d\theta + g\, d\phi + c\, d\rho) \wedge (f'\, d\rho \wedge d\theta + g'\, d\rho \wedge d\phi)$$

$$= (f g' - f' g)\, d\rho \wedge d\phi \wedge d\theta > 0 \iff f g' - f' g =: \Delta(\rho) > 0.$$

$\Rightarrow$ λ^c is ctct $\forall\, c \in \mathbb{R}$. $\Rightarrow$ so is $\frac{1}{c}\alpha_k + d\rho$.

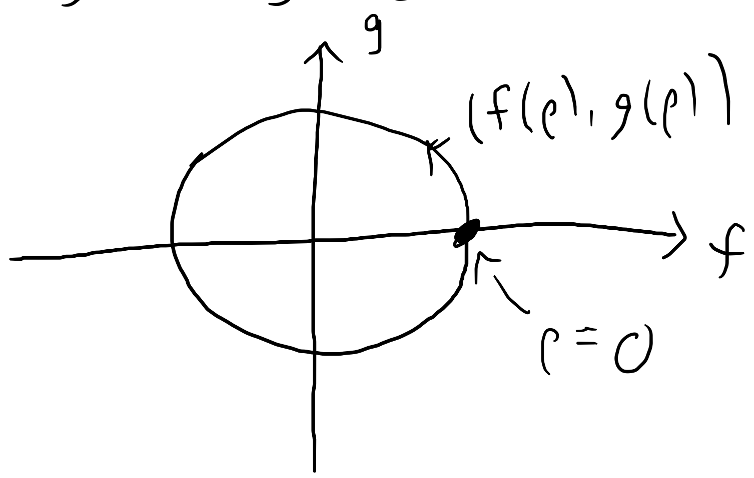
$\Rightarrow$ $\exists$ smooth deformation of ctct strs. from ξ_k to $\ker d\rho$

$\Rightarrow$ $\exists$ smooth deformation of ctct strs. btwn ξ_k & ξ_ℓ $\forall\, k, \ell \in \mathbb{N}$.

ALERT: $\ker d\rho$ is <u>not</u> a ctct str.

<u>thm</u> (to be proved ~ week 10): $(\mathbb{T}^3, \xi_k)$, $(\mathbb{T}^3, \xi_\ell)$ are <u>not</u> contactomorphic for $k \neq \ell$. (but they <u>are</u> homotopic)

$\alpha = f(\rho)\, d\theta + g(\rho)\, d\phi$ on $\mathbb{T}^3$, $\quad D := fg' - f'g > 0$



$d\alpha = f'\, d\rho \wedge d\theta + g'\, d\rho \wedge d\phi$

$\Rightarrow R_\alpha = \frac{g'}{D} \partial_\theta - \frac{f'}{D} \partial_\phi .$

<u>Assume</u> at <u>$\rho = 0$</u>: $f > 0$, $g = 0$, $f' = 0$, $g' > 0$,

On $\{\rho = 0\} \cong \mathbb{T}^2$, $R_\alpha = \frac{1}{f(\rho)} \partial_\theta \Rightarrow$ this 2-torus is foliated by an S^1-family of orbits w/ period $T := f(0) > 0$.

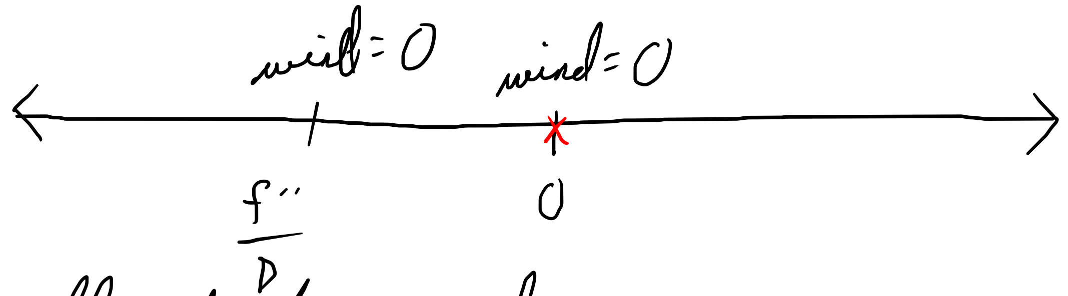
1-param. family $\Rightarrow \ker A_\gamma$ contains <u>at least</u> a 1-dim. space of sections pointing along this 2-torus.

<u>EX</u>: For each of these orbits γ $\exists$ a natural choice of cpx str. $J : \xi \to \xi$, $\exists$ a triv. of $\gamma^*\xi$ s.t. A_γ becomes

$$-i\partial_t + \begin{pmatrix} \frac{f''(0)}{D(0)} & 0 \\ 0 & 0 \end{pmatrix}.$$

Assume $f''(0) < 0$.

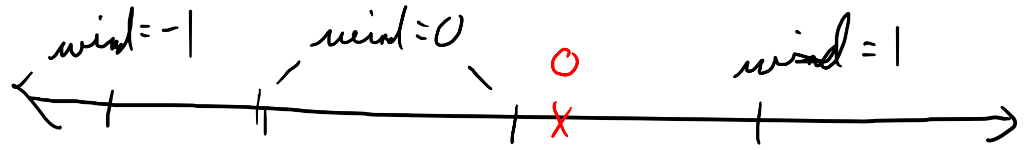
$\sigma(A_\gamma)$



<u>Claim</u>: After any pert. of α making all orbits nondeg., all orbits in some nbhd of $\{\rho = 0\}$ have (in our chosen triv.)

$\mu_{CZ} = 0$ (pos. hyp.) or 1 (elliptic).

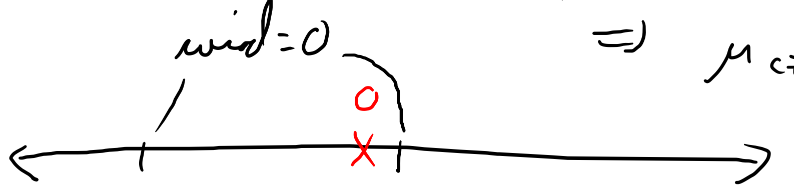
<u>case 1</u>:



$\alpha_- = 0, \quad \alpha_+ = 1 \Rightarrow p = 1$

$\Rightarrow \mu_{CZ} = 2\alpha_- + p = 1.$

<u>case 2</u>:



$\alpha_- = \alpha_+ = 0 \Rightarrow p = 0, \quad \mu_{CZ} = 0.$