rk: In fin. dims., maps are smooth until proven non-smooth.

In ∞-dims,

ex: H a real Hilbert space: $f: H \longrightarrow \mathbb{R} : x \longmapsto \|x\|^2$

$$H \xrightarrow{\text{linear}} H \times H \xrightarrow{\text{bilinear}} \mathbb{R} \implies C^\infty.$$
$$x \longmapsto (x, x)$$
$$(x,y) \longmapsto \langle x, y \rangle$$

False on Banach spaces in general! $\implies \nexists$ bump fns., P.O.U.s etc.

Q: When does a space of maps $N^n \longrightarrow M^m$ have a Banach mfd str.?

cpt? (can be generalized)

refs: Eliasson
Palais

Assume N a cpt mfd of class C^r, $1 \le r \le \infty$.

__defn__: A __section__ __functor__ $\mathcal{S}$ assigns to each VB $E \longrightarrow N$ of class C^r a Banach space $\mathcal{S}(E)$ of (equivalence classes a.e.) of sections $s : N \to E$ s.t. for 2 bundles $E, F \to N$, $\exists$ contin. linear incl.

$$C^r(\mathrm{Hom}(E,F)) \xrightarrow{\Phi} \underbrace{\mathcal{L}}_{\text{bdd linear maps}}(\mathcal{S}(E), \mathcal{S}(F)), \qquad \Phi(A)\eta := A\eta.$$

__ex__: $\mathcal{S} = C^k$ is a section functor if $k \le r$.

__ex__: $\mathcal{S} = W^{k,p}$ " " if $k \le r$:

$\forall$ $A \in C^r(\mathrm{Hom}(E,F))$, $\eta \in W^{k,p}(E)$, $A\eta \in W^{k,p}(F)$

& $\|A\eta\|_{W^{k,p}} \le c\|A\|_{C^r} \cdot \|\eta\|_{W^{k,p}}$.

<u>defn</u>: A ser. fnct. $\mathcal{S}$ is a <u>manifold</u> <u>model</u> if:

(1) $\forall\ E \to N$, $\exists$ contin. linear incl. $\mathcal{S}(E) \hookrightarrow C^0(E)$.

(2) $\forall\ E, F \to N$, $\exists$ contin lin. incl. $\mathcal{S}(\mathrm{Hom}(E,F)) \longrightarrow \mathcal{L}(\mathcal{S}(E), \mathcal{S}(F))$,

i.e. $\forall\ A \in \mathcal{S}(\mathrm{Hom}(E,F))$, $\eta \in \mathcal{S}(E)$, we have $A\eta \in \mathcal{S}(F)$ &

$\|A\eta\|_{\mathcal{S}} \leq c\|A\|_{\mathcal{S}} \cdot \|\eta\|_{\mathcal{S}}$ ("Banach algebra property")

(3) $\forall\ E, F \to N$, $\mathcal{O} \overset{\text{open}}{\subseteq} E$ intersecting every fiber &

$f : \mathcal{O} \to F$ a C^r fiber-pres. (but possibly nonlinear) map,

$\eta \in \mathcal{S}(\mathcal{O}) := \{\eta \in \mathcal{S}(E) \mid \eta(N) \subseteq \mathcal{O}\} \Rightarrow f \circ \eta \in \mathcal{S}(F)$, & the map

$\mathcal{S}(f) : \mathcal{S}(\mathcal{O}) \longrightarrow \mathcal{S}(F) : \eta \longmapsto f \circ \eta$ is contin. (<u>ex</u>: for $W^{k,p}$, "C^k-continuity")

<u>ex</u>: For $\dim N = n$, $W^{k,p}$ is a mfd model iff $kp > n$.

<u>main lemma</u> (last week): If $\mathcal{S}$ is a mfd model & $f : \mathcal{O} \to F$ as above is of class C^{r+s} restricted to each fiber, for some $s \in \mathbb{N}$.

Then $\mathcal{S}(f) : \mathcal{S}(\mathcal{O}) \to \mathcal{S}(F)$ is of class C^s,

$$d\mathcal{S}(f) = \mathcal{S}(\underbrace{d_2 f}_{\text{derivs. of } f \text{ in fiber directions}})$$

$\boxtimes$

<u>rk</u>: N cpt, then (1) $\Rightarrow$ $\mathcal{S}(\mathcal{O}) \overset{\text{open}}{\subseteq} \mathcal{S}(E)$.

(3) becomes more complicated if N is not cpt.

Fix a mfd M of class C^∞ w/ connection ∇.

Fix an open nbhd $\mathcal{D} \subseteq TM$ of the 0-section, s.t.

for the bndl proj. $\tau : TM \longrightarrow M$, $(\tau, \exp)\big|_{\mathcal{D}} : \mathcal{D} \hookrightarrow M \times M$

is a diffeo onto its image.

<u>main thm</u>: For $\mathcal{S}$ a mfd model,

$$\mathcal{S}(N,M) := \{ \exp_f h : N \to M \mid f \in C^\infty(N,M),\ h \in \mathcal{S}(f^*\mathcal{D}) \}$$

$$(\text{i.e. } \forall\ x \in N,\ h(x) \in \mathcal{D} \ \text{ \& } \ h \in \mathcal{S}(f^*TM))$$

is a smooth Banach mfd, with charts given by inverse of

$$\mathcal{S}(f^*E) \overset{\text{open}}{\supseteq} \mathcal{S}(f^*\mathcal{D}) \longrightarrow \mathcal{S}(N,M) : h \longmapsto \exp_f h \quad \text{for each } f \in C^\infty(N,M).$$

<u>pf</u>: Transition maps are of the form $\mathcal{S}(g)$ for smooth fiber-pres. maps g. $\boxtimes$

$f : N \longrightarrow M$, $\quad h \in \mathcal{S}(f^*TM)$, $\quad h(x) = T_{f(x)}M$

$$(\exp_f h)(x) = \exp_{f(x)} h(x)$$

<u>EX</u>: For $\mathcal{S} = W^{k,p}$, $W^{k,p}(N,M) = W^{k,p}_{loc}(N,M) :=$

$$\{ f : N \to M \mid \text{in all choices of local coords., } f \in W^{k,p}_{loc} \}$$

Banach space bundles: $\pi : \underset{\text{top. space}}{E} \longrightarrow B = \text{Banach mfd}$ s.t. $B = \bigcup_{\alpha \in I} \mathcal{U}_\alpha$

for $\mathcal{U}_\alpha \overset{\text{open}}{\subseteq} B$ s.t. $\exists$ local trivs. $\Phi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \xrightarrow{\text{homeo}} \mathcal{U}_\alpha \times X_\alpha$

X_α a Banach space, & $\forall\ \alpha, \beta \in I$,

$$\begin{tikzcd}
 & \pi^{-1}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \arrow[dl, "\Phi_\alpha"', "\text{homeo}"] \arrow[dr, "\Phi_\beta", "\text{homeo}"'] & \\
(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times X_\alpha \arrow[rr, "\Phi_{\beta\alpha}"] & & (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times X_\beta
\end{tikzcd}$$

$\pi : E \longrightarrow B$ is smooth if $\forall\ \alpha, \beta$, $\Phi_{\beta\alpha}(x, v) = (x, g_{\beta\alpha}(x) v)$

for some smooth maps $g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \longrightarrow \mathcal{L}(X_\alpha, X_\beta)$.

Alert: That condition is stronger than just smoothness of $\Phi_{\beta\alpha}$.

rk: For $u \in \mathcal{S}(N,M)$, $T_u \mathcal{S}(N,M) = \mathcal{S}(u^* TM)$.

$u^* TM$ is in general a VB of class $\mathcal{S}$, not C^∞.

Banach alg. property $\Rightarrow$ $\mathcal{S}(u^* TM)$ is well-def'd.

thm: For a $\mathcal{S}$ a mfd model & $\mathcal{T}$ a section functor s.t. $\exists$

contin. incl. $\mathcal{S}(\mathrm{Hom}(E,F)) \longrightarrow \mathcal{L}(\mathcal{T}(E), \mathcal{T}(F))$,

$\exists$ a smooth VB $\mathcal{E} \longrightarrow \mathcal{S}(N,M)$ with fibers $\mathcal{E}_u := \mathcal{T}(u^* TM)$.

ex: Can take $W^{k,p} =: \mathcal{S}$ for $kp > n$, $\mathcal{T} = W^{\ell,p}$ for any $\ell \le k$,

since $\exists$ contin product pairing $W^{k,p} \times W^{\ell,p} \longrightarrow W^{\ell,p}$.

e.g. $\mathcal{E}_u = W^{k-1,p}(u^* T\hat{W})$ as bundle over $\mathcal{B} := W^{k,p}(\Sigma, \hat{W})$.