some useful tools in dim. 4

<u>rk</u>: If J is explicit enough for $\bar\partial_J u = 0$ to be solvable (explicitly), then one cannot assume J is "generic".

<u>"automatic $\pitchfork$":</u>

<u>thm</u>: Spse $u : (\dot\Sigma_g, j) \looparrowright (\hat W^4, J)$ is immersed & $\mathrm{ind}(u) > 2g + \#\Gamma_0 - 2$ (where $\Gamma_0 := \{ z \in \Gamma \mid \mu_{CZ}(\gamma_z) \text{ is even} \}$). Then u is Fredholm regular.

<u>ex</u>: $W^4 = S^2 \times \Sigma_g$ closed, $J := i \oplus j$ for some $j \in \mathcal{J}(\Sigma_g)$.

Then $\forall$ $w \in \Sigma_g$, $u_w(z) := (z, w)$ is a J-hol. sphere

$\zeta \in S^2$, $v_\zeta(z) := (\zeta, z)$ is a J-hol. curve $(\Sigma_g, j) \to (W, J)$.

Both have trivial normal bndls, so $c_1(u_w^* TW) = c_1(TS^2) + c_1(N_{u_w}) = 2$

$c_1(v_\zeta^* TW) = c_1(T\Sigma_g) + c_1(N_{v_\zeta}) = 2 - 2g$

$\Rightarrow \mathrm{ind}(u_w) = (\overset{2}{\overset{\shortparallel}{n}} - 3)\chi(S^2) + 2c_1(u_w^* TW) = 2.$

$\mathrm{ind}(v_\zeta) = (n-3)\chi(\Sigma_g) + 2(2-2g) = \chi(\Sigma_g) = 2 - 2g \leq 0$ if $g \geq 1$,

although $\exists$ a 2-param. of curves near v_ζ. $\Rightarrow v_\zeta$ not regular.

$\Rightarrow J$ is not "generic".

But $\mathrm{ind}(u_w) > 2(0) - 2 \Rightarrow u_w$ <u>is</u> regular.

__fact__: For $u : \dot{\Sigma} \hookrightarrow \hat{W}$, $\quad u^*T\hat{W} = T\dot{\Sigma} \oplus \underset{\substack{\text{normal} \\ \text{bndl}}}{\underset{\sim}{N_u}}$ $\rightsquigarrow$ $D_u = \begin{pmatrix} D_u^T & D_u^{TN} \\ D_u^{NT} & D_u^N \end{pmatrix}$

s.t. u is regular iff $D_u^N : W^{k,p,\delta}(N_u) \to W^{k-1,p,\delta}(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, N_u))$ is surjective.

Since $2n=4$, N_u is a cpx __line__ bndl.

A criterion for CR-ops. on line bndls

(1) __Case Σ closed__: $E \to \Sigma$ line bndl, D a CR-op. on E.

D surj $\iff$ D^* inj, $\quad \mathrm{ind}(D^*) = -\mathrm{ind}(D)$, $\quad D^*$ is a CR-op. on a bndl $\hat{E}$,

$-\mathrm{ind}(D) = \mathrm{ind}(D^*) = \chi(\Sigma) + 2c_1(\hat{E}) \Rightarrow 2c_1(\hat{E}) = -\chi(\Sigma) - \mathrm{ind}(D)$.

If $c_1(\hat{E}) < 0$, $\eta \neq 0 \in \ker D^*$ has $c_1(\hat{E}) = \#\,\eta^{-1}(0) \geq 0$ by sim. pos.,

contra. unless $\eta \equiv 0$, $\therefore D^*$ inj.

$-\chi(\Sigma) - \mathrm{ind}(D) < 0 \iff \mathrm{ind}(D) > 2g-2 \Rightarrow D$ surj.

(2) <u>punctured case</u> $\dot{\Sigma} = \Sigma \setminus \Gamma$

<u>asymptotic lemma</u> (HWZ + E. Mora + R. Siefring): Suppose D is a CR-op. on $E \to \dot{\Sigma}$ asymp. to nondeg. asymp. ops. $\{A_z\}_{z\in\Gamma^\pm}$. If $\eta \in W^{k,p}(E)$ & $D\eta = 0$, $\eta \neq 0$ then in hol. cyl. coords $(s,t) \in Z_\pm$ & asymp. triv. near each pt. $z \in \Gamma^\pm$,

$\eta(s,t) = e^{\lambda s}(f_\lambda(t) + r(s,t))$ for some e-fn. f_λ of A_z with

$A_z f_\lambda = \lambda f_\lambda$, $\pm\lambda < 0$, & $|r(s,t)| \to 0$ as $s \to \pm\infty$.

<u>Recall from Week 3</u>: $\sigma(A_z) \to \mathbb{Z} : \lambda \longmapsto \mathrm{wind}^\tau(f_\lambda)$ is a well-def'd fn.

taking every value twice (counting multiplicity), & monotone increasing.

$\mu_{CZ}^\tau(A_z) = 2\alpha_-^\tau(A_z) + p(A_z) = 2\alpha_+^\tau(A_z) - p(A_z)$ w/

$\alpha_+^\tau(A_z) := \min\{\mathrm{wind}^\tau(f_\lambda) \mid \lambda > 0\}$, $\alpha_-^\tau(A_z) := \max\{\mathrm{wind}^\tau(f_\lambda) \mid \lambda < 0\}$.

<u>defn</u>: The <u>adjusted</u> <u>1st Chern #</u> of $(E, \{A_z\})$ is

$$c_1(E,\{A_z\}) := c_1^\tau(E) + \sum_{z\in\Gamma^+} \alpha_-^\tau(A_z) - \sum_{z\in\Gamma^-} \alpha_+^\tau(A_z) \in \mathbb{Z}$$

(τ := arb. choice of asymp. triv. — total is indep. of choice)

Given $\eta \in \Gamma(E)$ w/ fin.-many zeroes, let

$Z(\eta) := \sum_{z\in\eta^{-1}(0)} \mathrm{ord}(\eta; z) \in \mathbb{Z}$, not indep. of the choice of η (unless $\Gamma = \emptyset$)

$$Z_\infty(\eta) := \sum_{z\in\Gamma^+} \left[\alpha_-^\tau(A_z) - \mathrm{wind}^\tau(\eta \text{ near } z)\right] + \sum_{z\in\Gamma^-} \left[\mathrm{wind}^\tau(\eta \text{ near } z) - \alpha_+^\tau(A_z)\right].$$

<u>cor</u> (of sim. princ. + asymptotic lemma): If $D\eta = 0$ & $\eta \neq 0$, then

$Z(\eta)$ & $Z_\infty(\eta) \geq 0$. $Z(\eta) = 0$ iff η is never zero, $Z_\infty(\eta) = 0$ iff η attains the "extremal" possible winding at every puncture.

Informally, $Z_\infty(\eta)$ = "# zeroes of η hidden at ∞".

<u>EX</u>: $Z(\eta) + Z_\infty(\eta) = c_1(E, \{A_z\})$ $\forall$ $\eta \in \Gamma(E)$ w/ fin. many zeroes.

<u>cor</u>: If $c_1(E, \{A_z\}) < 0$, then D is inj. $\boxed{\text{ч}}$

<u>pf of thm</u>: Need to show, if $\underset{\text{ind}(D_u^N)}{\overset{\shortparallel}{\text{ind}(u)}} > 2g - 2 + \#\Gamma_0$, (let's abbreviate: $E := N_u$, $D := D_u^N$)

then $c_1(\hat{E}, \{\hat{A}_z\}) < 0$ ($\Rightarrow D^+$ is inj), where D^+ is a CR-op. on $\hat{E}$

asymp. to $\{\hat{A}_z\}_{z\in\Gamma}$.

Recall: $\text{ind}(D) = \chi(\dot{\Sigma}) + 2c_1^\tau(E) + \sum_{z\in\Gamma^+} \mu_{CZ}^\tau(A_z) - \sum_{z\in\Gamma^-} \mu_{CZ}^\tau(A_z)$

$= 2 - 2g - \#\Gamma + 2c_1(E, \{A_z\}) - 2\sum_{z\in\Gamma^+} \alpha_-^\tau(A_z) + 2\sum_{z\in\Gamma^-} \alpha_+^\tau(A_z)$

$+ \sum_{z\in\Gamma^+} \left[2\alpha_-^\tau(A_z) + p(A_z)\right] - \sum_{z\in\Gamma^-} \left[2\alpha_+^\tau(A_z) - p(A_z)\right]$

$= 2 - 2g - \#\Gamma_0 + 2c_1(E, \{A_z\})$.

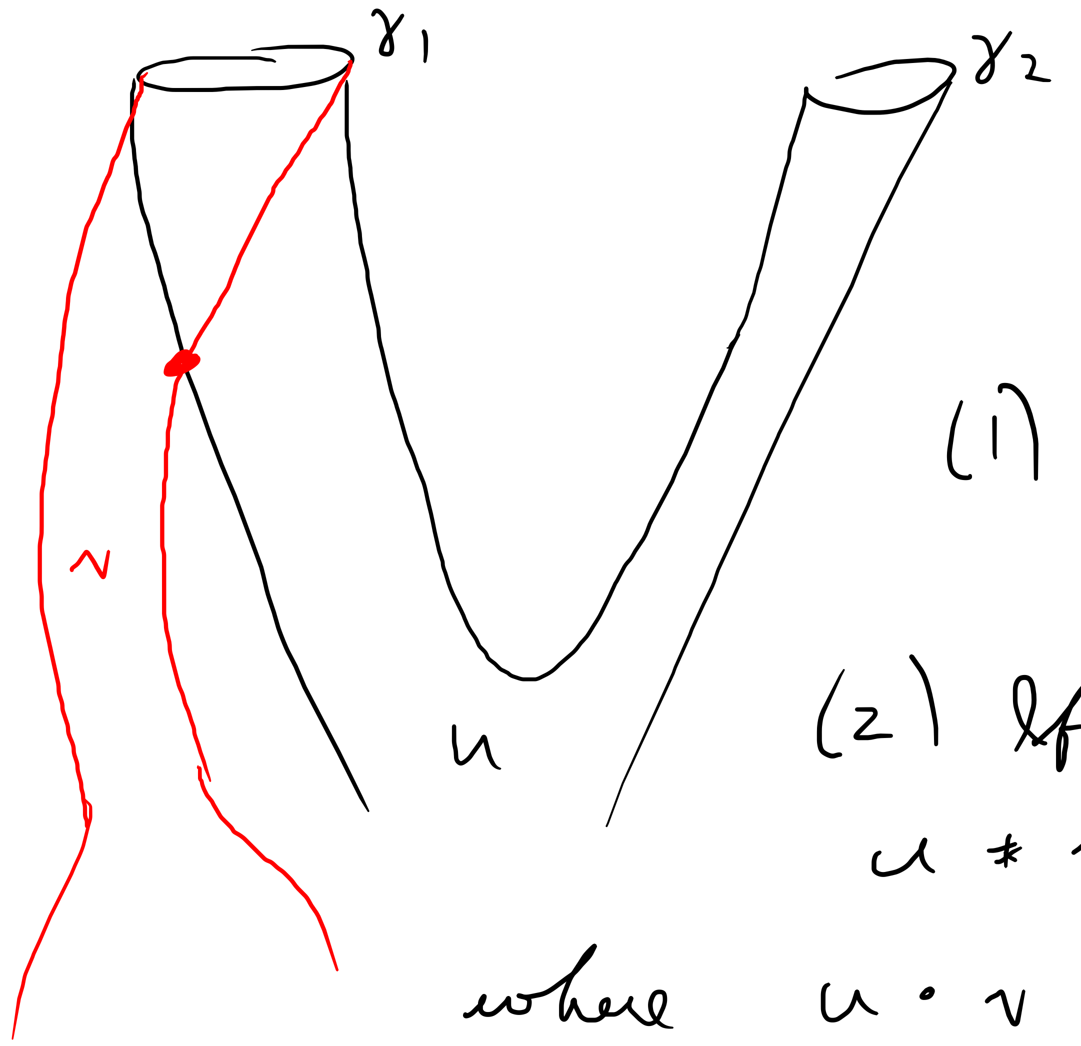
Apply this to D^+: $2c_1(\hat{E}, \{\hat{A}_z\}) = \text{ind}(D^+) - 2 + 2g + \#\Gamma_0$

$= -\text{ind}(D) - 2 + 2g + \#\Gamma_0 < 0 \iff$ stated criterion. $\boxtimes$

sketch of intersection theory (R. Siefring)

Goal: for asymp. cyl. curves u, v in $\hat{W}^4$, defn. an algebraic count of intersections that is htpy inv't.

Problem: Under homotopies, ints. can escape to ∞.



thm: $\forall$ asymp. cyl. maps $u : \dot{\Sigma} \to \hat{W}^4$,
$v : \dot{\Sigma}' \to \hat{W}^4$,

one can associate a number $u * v \in \mathbb{Z}$, s.t.

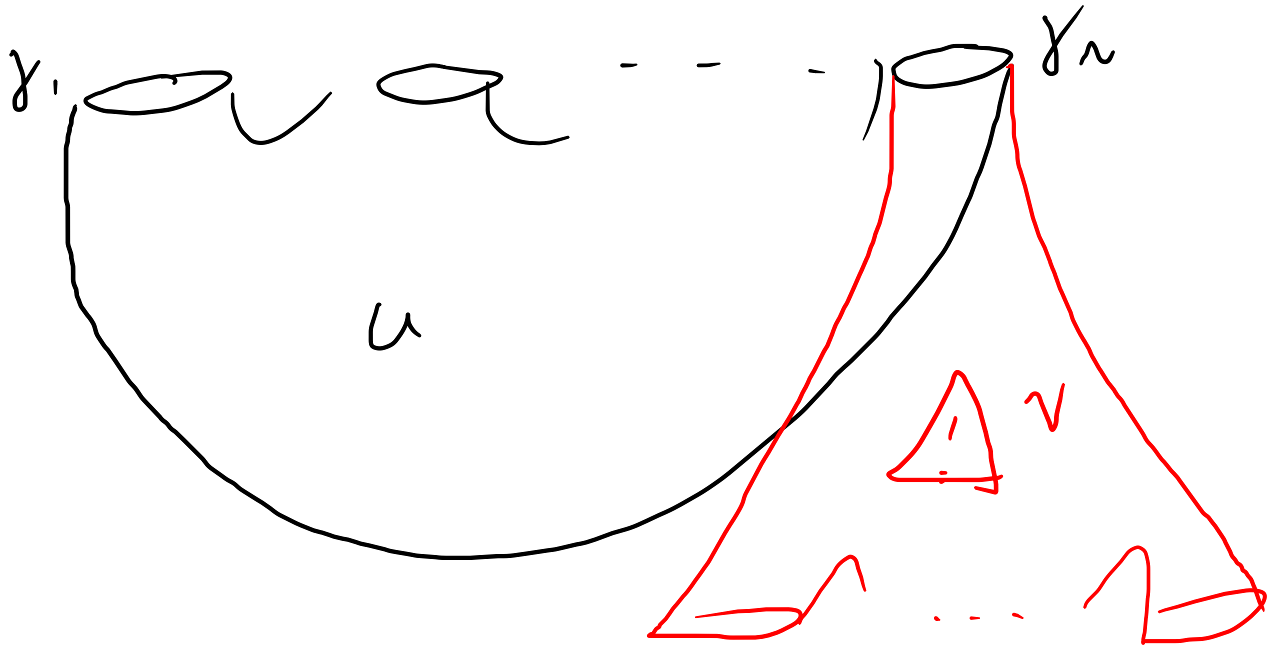
(1) Depends only on the rel. homol. classes $[u], [v]$ & their asymptotic orbit. ($\Rightarrow$ htpy inv't)

(2) If both are J-hol. & images are not identical, then
$$u * v = u \cdot v + i_\infty(u, v)$$
where $u \cdot v :=$ algebraic count of ints. (which satisfies
$$0 \leq \left| \{ (z, \zeta) \in \dot{\Sigma} \times \dot{\Sigma}' \mid u(z) = v(\zeta) \} \right| \underset{\uparrow = \text{ iff } u \pitchfork v}{\leq} u \cdot v$$

$i_\infty(u, v) :=$ "alg. count of hidden ints. at ∞" ≥ 0

$= 0$ iff "all relative asymptotic e-fns describing approach of u to v near ∞ have extremal winding."

<u>application</u>: Suppose $(M^3, \xi = \ker\alpha)$ a cont. 3-mfd w/ nondeg. Reeb embedded orbits $\gamma_1, \dots, \gamma_N$ s.t. $M \setminus (\gamma_1 \cup \dots \cup \gamma_N)$ is foliated by projections to M of J-hol. curves $u : (\dot\Sigma, j) \longrightarrow (\mathbb{R} \times M, J)$ w/ pos. ends asymp. to $\gamma_1, \dots, \gamma_N$ (& no neg. ends), & all ends approach orbits w/ extremal winding. Then all other curves v w/ pos. ends asymp. to a subset of these & <u>any</u> neg. ends belong to this foliation.



<u>pf</u>: If v not part of the foliation, then has isolated ints. w/ some curve u in the foliation,

$$0 < u \cdot v \le u * v \overset{\text{htpy invariance}}{=} u * v'$$
$$= u * (\mathbb{R} \times \gamma_N)$$
$$= i_\infty(u, \mathbb{R} \times \gamma_N)$$
$$\overset{\text{(extremal winding)}}{=} 0. \quad \blacksquare$$