Symplectic homology I

Agenda: Setup, No-Escape Lemma, Morse-Bott moduli spaces, Def. of SH

Bib.: Bourgeois/Oancea: – "SH, aut. Ham. and Morse-Bott moduli spaces"
– "An ex. seq. for Contact Hom. & SH"
Seidel: – "A biased view of SH"
Ritter: – "TQFT structure on SH"
Cieliebak/Oancea: – "SH and Eilenberg-Steenrod axioms"
F.: – "On manifolds with so many fillable contact str."

Setup

- (V, λ) is a Liouville domain if
  - V is a $2n$-dim compact mfld with bdry $\partial V = \Sigma$
  - λ is a 1-form on V, s.t. $d\lambda = \omega$ is symplectic
  - $\lambda \wedge (d\lambda)^{n-1}\big|_{T\Sigma}$ is a volume form on Σ, i.e. $\lambda\big|_{T\Sigma} =: \alpha$ is a contact form
- Y is the Liouville vf on V, d. by $\omega(Y, \cdot) = \lambda$

- (V, λ) is a <u>Weinstein domain</u>, if there exists Morse Fct $H : V \to \mathbb{R}$
  s.t. (- H is bounded from below)
  - Y is gradient like for H, i.e. $dH(Y) \geq 0$ and $dH_p(Y) = 0 \iff p \in \mathrm{crit}(H)$)
- for α contact form on Σ, $R = R_\alpha$ <u>Reeb vf</u> on Σ:
  - $\alpha(R_\alpha) = 1$, $- d\alpha(R_\alpha, \cdot) = 0$
- spectrum of α $\mathrm{Spec}(\Sigma, \alpha) = P(\alpha) = \{ T \in \mathbb{R}^+ \mid \exists \text{ closed } R_\alpha\text{-orbit with period } T \}$
- completion $(\hat{V}, \hat{\lambda})$ of (V, λ) is
  $\hat{V} := V \cup [0, \infty) \times \Sigma$, $\hat{\lambda} := \begin{cases} \lambda & \text{on } V \\ e^r \alpha & \text{on } [0,\infty) \times \Sigma \end{cases}$

<u>Note</u>: - $(\mathbb{R} \times \Sigma, e^r \cdot \alpha) \xhookrightarrow{\text{symp}} (\hat{V}, \hat{\lambda})$ via the flow φ_Y,
as $\mathcal{L}_Y \lambda = \lambda$, $\mathcal{L}_Y \omega = \omega$
- For (V, λ, H) Weinstein, then can also extend H to $(\hat{V}, \hat{\lambda})$.

A <u>Hamiltonian</u> is a S^1-family of fct, i.e.
$H : V \times S^1 \to \mathbb{R}$, <u>Ham. vf</u> X_{H_t} defined by
$dH_t = \omega(\cdot, X_{H_t}) \quad \forall t \in S^1$.

- action functional for H_t on loops $x : S^1 = \mathbb{R}/\mathbb{Z} \to \hat{V}$

$$\mathcal{A}^H(x) := \int_0^1 x^*\lambda - \int_0^1 H_t(x(t))\,dt$$

- $\mathcal{P}(H) = \{ x : S^1 \to \hat{V} \mid \dot{x}(t) = X_{H_t}(x(t)) \} = \mathrm{Crit}\,\mathcal{A}^H$
- J_t <u>S^1-family of a.c.s.</u> on $\hat{V}$, ω-comp., i.e. $\omega(\cdot, J_t \cdot)$ is Riemannian metric $\forall t \in S^1$.
- $u : \mathbb{R} \times S^1 \to \hat{V}$ <u>Floer cylinder / sol. to Floer eq.</u>

$$(1(J)) \quad \partial_s u - \nabla \mathcal{A}^H(u) = \partial_s u + J_t(\partial_t u - X_{H_t}) = 0$$

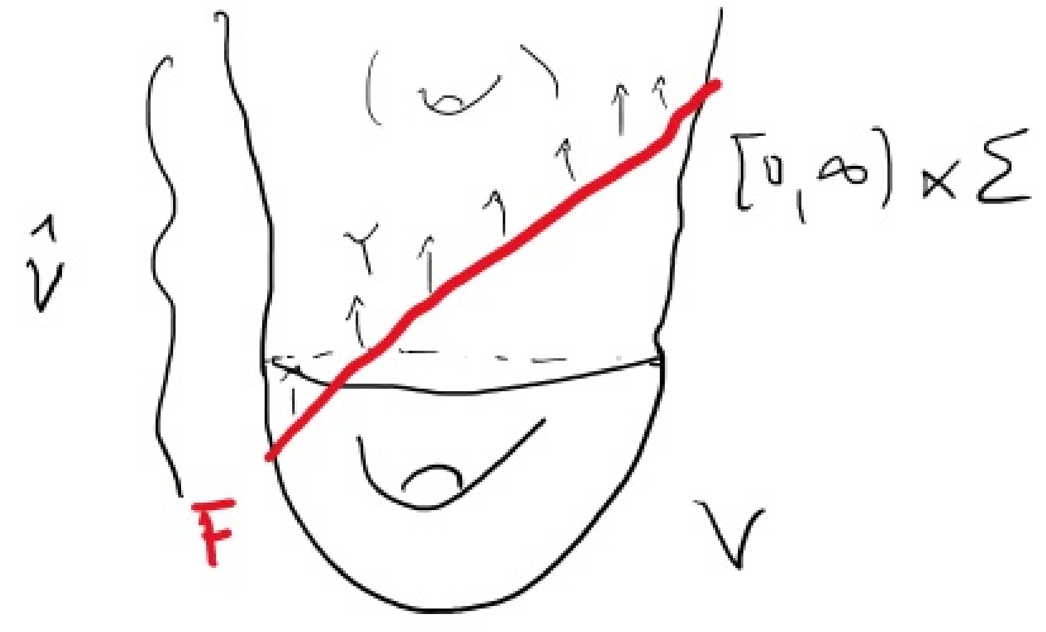
$$\iff \left(Du - X_{H_t} \otimes dt\right)^{0,1} = 0$$

Du is a 1-form on $\mathbb{R} \times S^1$ with values in $T\hat{V}$

$\beta^{0,1} := \frac{1}{2}(\beta + J\beta j)$, where j is a.c.s. on $\mathbb{R} \times S^1$, $j\,\partial_s = \partial_t$

No-Escape-Lemma

goal: Given a hypersurface F transverse to Y in $\hat{V}$ and a Floer sol. u with $\lim_{s\to\pm\infty} u$ below F, we want to show that u stays below F for all (s,t)



The flow of Y provides a tubular nghd around F
$$(-\varepsilon,\varepsilon)\times F \hookrightarrow \hat{V} \text{ via}$$
$$(r,p) \longmapsto \varphi_Y^r(p)$$

Lemma 1 (No escape Lemma) (Ritter, F.)

- Assume $V_0 \subset \hat{V}$ is compact with bdry $\partial V_0 = F$ transverse to Y
- $H_t^s : \hat{V} \to \mathbb{R}$ is near F of the form $H_t^s(r,p) = h_s(e^r)$ $\left(= a_s e^r + b_s\right)$ and $\partial_s\left(H_t^s - h_s(1) + h_s'(1)\right) \leq 0$ on $\hat{V} \setminus V_0$
- J is of contact type along F, meaning $J^* \hat{\lambda} = d e^r$, and preserves the contact str.
- $S \subset \mathbb{R}\times S^1$ compact (Riemannian) surface with smooth bdry

- $u : S \to V$ is sol to Floer eq. and

  $u(\partial S) \subset F$ and $r(u(s,t)) \geq 0$

Then $u(S) \subset F$.

<u>Proof</u>

$$0 \leq E_S(u) := \int_S \|\partial_s u\|^2 ds \wedge dt = \int_S d\lambda(\partial_s u, J\partial_s u)\, ds \wedge dt$$

$$= \int_S d\lambda(\partial_s u, \partial_t u - X_H)\, ds \wedge dt$$

$$= \int_S u^* d\lambda - dH_s(X_H)\, ds \wedge dt$$

$$= \int_S u^* d\lambda - d(H_s(u)dt) + \underbrace{(\partial_s H_s)(u)\, ds \wedge dt}_{\square}$$

$$\overset{\text{Stokes}}{=} \int_{\partial S} u^* \lambda - H_s(u) dt \quad + \quad \square$$

$$= \int_{\partial S} u^* \lambda - \lambda(X_H) dt + \big(h'_s(1) - h(1)\big) dt + \square$$

$$= \int_{\partial S} \lambda(Du - X_H \otimes dt) + \underbrace{\int_S \partial_s \big(-h(1) + h'_s(1) + H_s(u)\big)\, ds \wedge dt}_{\leq 0}$$

$$\leq \int_{\partial S} -\lambda\,(J\,(Du - X_H^s \otimes dt)_j\,)$$

$$= \int_{\partial S} -de^r\,(Du - X_H \otimes dt)_j = \int_{\partial S} -de^r(Du)_j \quad ,\text{ as}$$

$r = 0$, i.e. F is a level set of H

If n is outer normal to ∂S, then jn orients ∂S, so

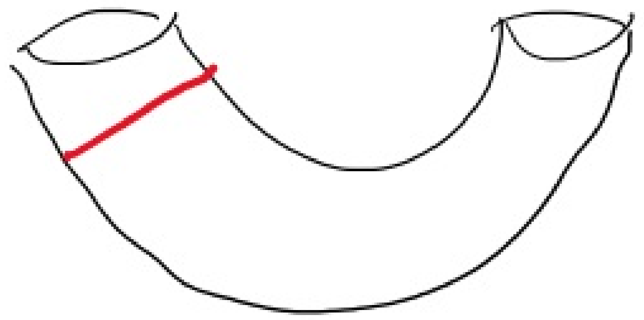
$$-de^r\,(Du)_j\,(jn) = -d(e^r \circ u)\,(-n) \leq 0,$$

as $e^r \circ u$ increases along the inward direction $-n$ of S

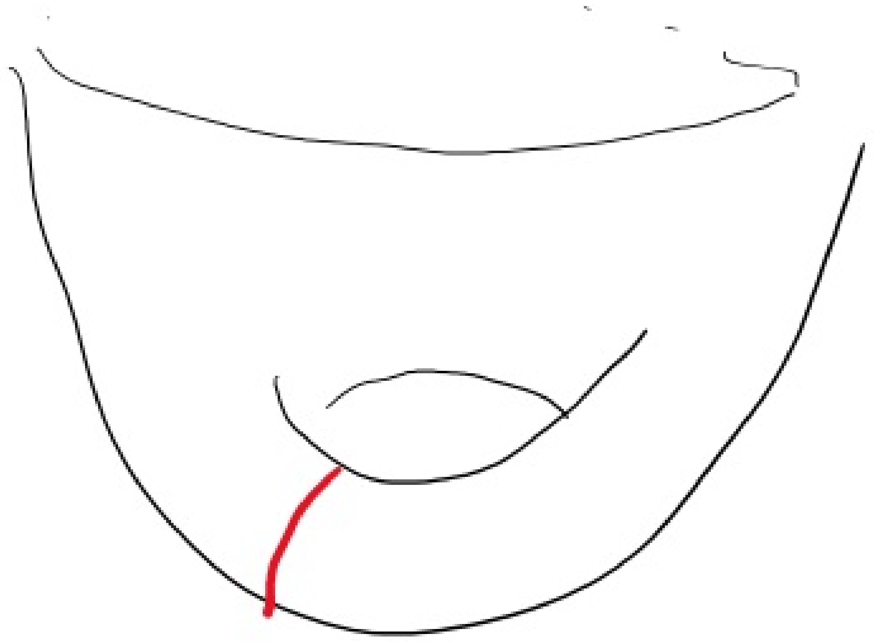
$$\Rightarrow \; 0 \leq E_S(u) \leq 0 \quad \Rightarrow \quad E_S(u) = 0 \quad \Rightarrow \partial_s u \equiv 0$$

$$\Rightarrow u(S) \subset F$$

<u>Cor.</u>: If $V_0 \subset \hat{V}$, $\partial V_0 = F$ and H_t^s are as above and J is of contact type on nbhd of F and $u : \mathbb{R} \times S^1 \to \hat{V}$ sol of (1)(J) with $\lim_{s \to \pm\infty} u \subset V_0$, then $u(s,t) \in V_0 \;\; \forall (s,t)$.



V



Morse-Bott moduli spaces

Consider Hamiltonians $H : \hat{V} \times S^1 \to \mathbb{R}$, s.t.

(limit) $\quad H_t(r,p) = a e^r + b \quad$ on $[R,\infty) \times \Sigma \subset \hat{V}$ for some $R \geq 0$

and $a \notin \mathrm{Spec}(\Sigma, \alpha)$

or $\quad H^s_t(r,p) = a_s e^r + b_s \quad$ with $\partial_s a_s \leq 0$

$\Rightarrow$ Lemma 1 is satisfied

(and $H^s_t = H^-_t$ for $s \ll 0$, and $H^s_t = H^+_t$ for $s \gg 0$)

Non-degeneracy : $\quad x \in \mathcal{P}(H)$ is <u>non-deg</u> if

$$(*) \qquad \operatorname{Det}\left(\mathbb{1} - \underline{D\varphi^{1}_{X_H}(x(0))}\right) \neq 0$$

In general, this holds only for time-dep. or C^2-small Hamiltonians.

Otherwise: If H is of the form $H(r,p) = h(e^r)$ on $[0,\infty) \times \Sigma \subset \hat{V}$, then

$$dH_{(r,p)} = h'(e^r)\, de^r = h'(e^r) \cdot d\lambda(\cdot, R_\alpha) = d\lambda(\cdot, h'(e^r) \cdot R_\alpha)$$

$$\Rightarrow X_H(r,p) = h'(e^r) \cdot R_\alpha(p)$$

$\Rightarrow$ all 1-per. orbits of a time indep. X_H come at least in S^1-families, param. by the starting point $x(0)$

$\Rightarrow$ they are not non-deg!

To ways to deal with this situation:

- either time-dep. perturb H
- or do Morse-Bott constr.

(Bourgeois Oancea + Cieliebak, Floer, Hofer, Wysocki "App. of SH II")

↳ Assume that H satisfies Morse Bott assumption.

SH

. or do Morse - Bott constr.

↳ Assume that H satisfies Morse Bott assumption, i.e. $\mathcal{A}^H$ is Morse - Bott :

(MB) $P(H)$ is a discrete union of manifolds N^i, s.t. $\forall x \in N^i$ holds:

$$\ker\left(\mathbb{1} - D\varphi_{X_H}^{-1}(x(0))\right) = T_x N^i$$

<u>Fact</u> If $C_\pm \subset P(H)$ are connected comp. then

$$\widehat{\mathcal{M}}_J(C_-, C_+) = \left\{ u \text{ sol of } (1)(J) \,\middle|\, \lim_{s \to \pm\infty} u \subset C_\pm \right\}$$

is for generic J a smooth manifold