The Legendrian Homology Algebra, Three Differentials and Linearization

Klaus Mohnke

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R: closed Reeb orbits are **non-degenerate**: $\gamma : [0, T] \to Y$ closed flow–line of R, $T > 0$, $\gamma(0) = \gamma(T)$,

$$\det(d_{\gamma(0)}\Phi^R_T\big|_{\xi_{\gamma(0)}} - \mathrm{id}_{\cancel{T_{\gamma(0)}Y}\ \xi_{\gamma(0)}}) \neq 0.$$

$\Lambda_1, \ldots, \Lambda_k$: Reeb chords are **non-degenerate**: $\gamma : [0, T] \to Y$ $\underline{T>0}$
flow-line of R, $\gamma(0) \in \Lambda_{j_0}, \gamma(T) \in \Lambda_{j_1}$. Then

$$d_{\gamma(0)}\Phi^R_T(T_{\gamma(0)}\Lambda_{j_0}) \pitchfork T_{\gamma(T)}\Lambda_{j_1}.$$

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$\Rightarrow$ closed Reeb orbits and Reeb chords are isolated.

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$\mathcal{C}_{ij}$... set of all Reeb chords connecting $\overset{\Lambda_i}{\cancel{\mathcal{C}_i}}$ and $\overset{\Lambda_j}{\cancel{\mathcal{C}_j}}$
$\mathcal{C}_i := \mathcal{C}_{ii} \coprod \{e_i\}$, $\mathcal{C} := \coprod_{i,j} \mathcal{C}_{ij} \setminus \{e_i\}$

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$\mathbb{K}\langle\mathcal{C}\rangle$ is a left-right R-module via

$$e_i \cdot c = \delta_{ij} c \text{ for } c \in \mathcal{C}_{\ell j}$$
$$c \cdot e_i = \delta_{ij} c \text{ for } c \in \mathcal{C}_{j\ell}$$

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\begin{aligned}
LHA(\Lambda) &:= R \oplus \mathbb{K}\langle \mathcal{C} \rangle \oplus \mathbb{K}\langle \mathcal{C} \rangle \otimes_R \mathbb{K}\langle \mathcal{C} \rangle \oplus \mathbb{K}\langle \mathcal{C} \rangle \otimes_R \mathbb{K}\langle \mathcal{C} \rangle \otimes_R \mathbb{K}\langle \mathcal{C} \rangle \oplus \dots \\
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Choose *generic* compatible almost complex structure J on $\mathbb{R} \times Y$. The differential $d : LHA(\Lambda) \to LHA(\Lambda)$ is defined on chords $c \in \mathcal{C}$ via

$$d_{LHA} c := \sum_{|c| = \sum |b_j| + 1} \underbrace{n_{c; b_1 \ldots b_m}}_{\in \mathbb{Z}} b_1 \ldots b_m$$

where

$$n_{c; b_1 \ldots b_m} = \hat{\sharp} \mathcal{M}^Y_\Lambda(c; b_1, \ldots, b_m)/\mathbb{R},$$

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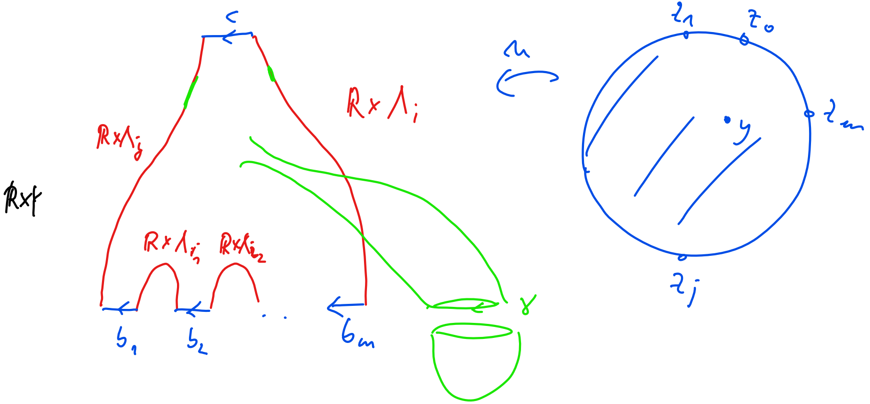
d_{LHA} is correctly defined: If $n_{c; b_1 \dots b_m} \neq 0$ then $b_1 \dots b_m$ are linearly decomposable.

$\mathcal{M}^{Y}_{\Lambda}(c; b_1, \dots, b_m)$



$L\mathbb{H}A(\Lambda)$ and $L\mathbb{H}A(\Lambda_i, \Lambda)$

We denote by $LHA(\Lambda_i; \Lambda)$ the differential graded subalgebra of $LHA(\Lambda)$ of words which begin and end on Λ_i.

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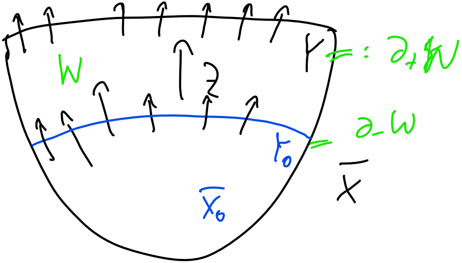
Proposition 4.3.: $d_{LHA}^2 = 0$. The homologies

$$L\mathbb{H}A(\Lambda) := H_*(LHA(\Lambda), d_{LHA}) \quad \text{and} \quad L\mathbb{H}A(\Lambda_i; \Lambda) := H_*(LHA(\Lambda_i\Lambda), d_{LH}$$

are independent of all choices $(\alpha, J, \ldots)$ and Legendrian isotopy invariants.



Setup

- $(\bar{X}, \omega, Z)$...Liouville domain, $\bar{X}_0 \subset \mathrm{int}\bar{X}$ subdomain such that Z points outward at $Y_0 = \partial \bar{X}_0$;
- $\bar{W} := \bar{X} \setminus \mathrm{int}\bar{X}_0$ ...Liouville cobordism, $\partial_- \bar{W} = Y_0, \partial_+ \bar{W} = Y := \partial \bar{X}$;
- W, X, X_0...completions of $\bar{W}, \bar{X}, \bar{X}_0$
- $L \subset W$...exact Lagrangian cobordism between Legendrians $\Lambda_- \subset Y_0$ and $\Lambda_+ \subset Y$.

Define homomorphism $F_L^W : LHA(\Lambda_+) \to LHA(\Lambda_-)$ on chords $c \in \mathcal{C}(\Lambda_+)$

$$F_L^W(c) := \sum_{|c| = \sum |b_j|} m_{c; b_1 \dots b_m} b_1 \dots b_m$$

where

$$m_{c; b_1 \dots b_m} = \hat{\sharp} \mathcal{M}_L^W(c; b_1, \dots, b_m),$$

where $b_1, \dots, b_m \in \mathcal{C}(\Lambda_-)$.

F_L^W

Proposition 4.4: (1) F_L^W is a homomorphism of graded algebras which is independent up to chain homotopy of all choices.
(2) If $L = \coprod_{j=0}^{k} L_k$ and $L_j \cap \partial_+ \bar{W} = \emptyset$ for $j > 0$
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$F_L^W(LHA(\Lambda_+)) \subset LHA(\Lambda_{0-}; \Lambda_-)$.

In particular, F_L^W induces homomorphismn

$$f_L^W : L\mathbb{H}A(\Lambda_+) \to L\mathbb{H}A(\Lambda_{0-}; \Lambda_-).$$

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Choose 0–cycle q representing a homology class $\mathbf{q} \in H_0(\Lambda)$:

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Define

$$LHA(\Lambda; \mathbf{q}) := R \oplus \mathbb{K}\langle \mathcal{C} \cup \{q\} \rangle \oplus \mathbb{K}\langle \mathcal{C} \cup \{q\} \rangle \otimes_R \mathbb{K}\langle \mathcal{C} \cup \{q\} \rangle \oplus \ldots$$

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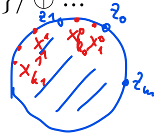

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The differential $d_{LHA;q}$ is defined on a chord $c \in \mathcal{C}(\Lambda)$ via

$$d_{LHA;q}c := \sum_{|c| = \sum |b_j| + 1 + k(n-2);\, k = k_0 + \dots + k_m} \underbrace{n_{c;b_1\dots b_m;k_0,\dots,k_m}}_{\in \mathbb{Z}} q^{k_0} b_1 q^{k_1} \dots q^{k_{m-1}} \underbrace{\dots}_{b_m q^{k_m}}$$

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$d_{LHA;q}q = 0$ and extend it to all words using graded Leibniz rule.

$L\mathbb{H}A(\Lambda; q)$

Proposition 4.5: $d^2_{LHA;q} = 0$. The homology

$$L\mathbb{H}A(\Lambda; q) = H_*(LHA(\Lambda; q), d_{LHA;q})$$

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$A = \mathbb{K}\langle b_1 q^{k_1} b_2 ... b_m q\rangle \subset (LHA(\Lambda; q), d_{LHA;q})$ unital subalgebra. Define

$$B := A/(q^2).$$

$d_{LHA;q}$ descends to d_B on B.

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Proposition 4.6: $L\mathbb{H}A(\Lambda_f; \Lambda \cup \Lambda_f) \cong H_*(B, d_B)$

The Cyclic Complex $LH^{cyc}(\Lambda)$

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$P : LHO^+(\Lambda) \to LHO^+(\Lambda)$ induced by

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$LH\cancel{O}^{cyc}(\Lambda) = LHO^{+}(\Lambda)/\mathrm{im}(1-P)$, d_{cyc} induced differential.

$LH\cancel{O}^{cyc}$ is not an algebra. If $w = c_1 \ldots c_\ell \in LHO^{+}(\Lambda)$ we denote $(w) \in LH\cancel{O}^{cyc}(\Lambda)$ and the **multiplicity** of (w) is the largest $k \in \mathbb{N}$ such that $(w) = (v^k)$ for some $v \in LHO^{+}(\Lambda)$.

Proposition 4.7: $d_{cyc}^2 = 0$ and

$$L\mathbb{H}^{cyc}(\Lambda) = H_*(LH^{cyc}(\Lambda), d_{cyc})$$

is independent of all choices and is Legendrian isotopy invariant of Λ.

The Complex $LH^{H_{0+}}$

$$LH^{H_{0+}}(\Lambda) := \overbrace{LHO^+}^{= LHO^+(\Lambda)}(\Lambda) \oplus \widehat{LHO}^+(\Lambda)$$

with grading shift $\widehat{LHO}^+(\Lambda) = LHO^+(\Lambda)[1]$.

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For $w = c_1 ... c_\ell \in LHO^{+}(\Lambda)$ we denote by $\check{w} := \check{c}_1 ... c_\ell \in LHO^{+}(\Lambda)$ and $\hat{w} := \hat{c}_1 c_2 ... c_\ell \in \widehat{LHO}^{+}(\Lambda)$ the corresponding monomials.

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Define $S : LHO^{+}(\Lambda) \to \widehat{LHO}^{+}(\Lambda)$ via

$$S(c_1 ... c_\ell) := \hat{c}_1 c_2 ... c_\ell + (-1)^{|c_1|} c_1 \hat{c}_2 ... c_\ell + ... + (-1)^{|c_1 ... c_{\ell-1}|} c_1 ... \hat{c}_\ell.$$

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The differential $d_{H_{0+}} : LH^{H_{0+}} \to LH^{H_{0+}}$ is given by

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$$d_{H_{0+}} := \begin{pmatrix} d_{LHO^+} & d_{MH_{0+}} \\ 0 & \hat{d}_{LHO^+} \end{pmatrix}$$

with

$$\hat{d}_{LHO^+}(\hat{c}w') = S(d_{LHO^+} c)w' + (-1)^{|c|+1} \hat{c}(d_{LHO^+} w')$$

for a chord c and $w' \in LHA(\Lambda)$ such that $\underline{cw'} \in LHO^+(\Lambda)$

The Complex $LH^{H_{0+}}$

$$LH^{H_{0+}}(\Lambda) := LHO^+(\Lambda) \oplus \widehat{LHO}^+(\Lambda)$$

with grading shift $\widehat{LHO}^+(\Lambda) = LHO^+(\Lambda)[1]$.

For $w = c_1 ... c_\ell \in LHO^+(\Lambda)$ we denote by $w := c_1 ... c_\ell \in LHO^+(\Lambda)$ and $\hat{w} := \hat{c}_1 c_2 ... c_\ell \in \widehat{LHO}^+(\Lambda)$ the corresponding monomials.

Define $S : LHO^+(\Lambda) \to \widehat{LHO}^+(\Lambda)$ via

$$S(c_1 ... c_\ell) := \hat{c}_1 c_2 ... c_\ell + (-1)^{|c_1} c_1 \hat{c}_2 ... c_\ell + ... + (-1)^{|c_1 ... c_{\ell-1}} c_1 ... \hat{c}_\ell.$$

The differential $d_{H_{0+}} : LH^{H_{0+}} \to: LH^{H_{0+}}$ is given by

$$d_{H_{0+}} := \begin{pmatrix} d_{LHO^+} & d_{MH_{0+}} \\ 0 & \hat{d}_{LHO^+} \end{pmatrix}$$

with

$$\hat{d}_{LHO^+}(\hat{c}w') = S(d_{LHO^+} c) w' + (-1)^{|c|+1} \hat{c}(d_{LHO^+} w')$$

for a chord c and $w' \in LHA(\Lambda)$ such that $cw' \in LHO^+(\Lambda)$ and

$w = c_1 \cdots c_\ell$

$$d_{MH_{0+}}(\hat{w}) = \hat{c}_1 c_2 ... c_\ell - c_1 c_2 ... \hat{c}_\ell$$

Proposition 4.8: $d_{Hot}^2 = 0$

$$LH^{Hot}(\Lambda) := H_*(LH^{Hot}(\Lambda), d_{Hot}))$$

is indep't of all choices & a Legendrian isotopy invariant.

Proposition 4.9 $\exists$ exact triangle

$$\begin{tikzcd}
LH^{Cyc}(\Lambda) \arrow[rr, "{[-2]}"] & & LH^{Cyc}(\Lambda) \arrow[dl, "{[1]}"] \\
& LH^{Hot}(\Lambda) \arrow[ul, "{[0]}"] &
\end{tikzcd}$$

? check degree shifts!