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Differentialgeometrie II
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Problem Set 1

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Problem 1

Assume $\pi : E \to B$ is a vector bundle of rank $m \geqslant 0$ over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ with bundle atlas $\{(\mathcal{U}_\alpha, \Phi_\alpha)\}_{\alpha \in I}$ and associated transition functions $\{g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{GL}(m, \mathbb{F})\}_{(\alpha,\beta) \in I \times I}$. Show that these satisfy the relations

$$g_{\alpha\alpha} \equiv \mathbb{1} \text{ on } \mathcal{U}_\alpha, \qquad \text{and} \qquad g_{\alpha\beta} g_{\beta\gamma} \equiv g_{\alpha\gamma} \text{ on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$$

for all $\alpha, \beta, \gamma \in I$. In particular, this implies $g_{\alpha\beta} \equiv g_{\beta\alpha}^{-1}$ on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ for all $\alpha, \beta \in I$.
Remark: One can show that for any open covering $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ of B and any collection of continuous matrix-valued functions $\{g_{\alpha\beta}\}_{(\alpha,\beta) \in I \times I}$ satisfying these algebraic relations, there exists a vector bundle with a bundle atlas for which these are the transition functions.

Problem 2

In the setting of Problem 1, assume $m \geqslant 1$ and $\mathbb{F} = \mathbb{R}$, and that the open covering $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ of B is chosen such that all of the sets $\mathcal{U}_\alpha$ and their double intersections $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ are connected whenever nonempty.[1] For each $(\alpha, \beta) \in I \times I$ with $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \varnothing$, define $\sigma_{\alpha\beta} \in \mathbb{Z}_2$ by

$$\sigma_{\alpha\beta} := \begin{cases} 0 & \text{if } \det g_{\alpha\beta} > 0 \text{ on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta, \\ 1 & \text{if } \det g_{\alpha\beta} < 0 \text{ on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta. \end{cases}$$

Prove: The bundle $E \to B$ is orientable if and only if there exists a function $I \to \mathbb{Z}_2 : \alpha \mapsto \tau_\alpha$ such that $\sigma_{\alpha\beta} = \tau_\alpha - \tau_\beta \bmod 2$ whenever $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \varnothing$.
Hint: Relate the numbers $\tau_\alpha \in \mathbb{Z}_2$ to the question of whether the corresponding local trivialization $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{R}^m$ preserves orientations of fibers.
Comment: Students with substantial knowledge of algebraic topology may sense that there is some cohomology going on in the background of this problem. In fact, this is the main step in the proof that a bundle $E \to B$ is orientable if and only if its first Stiefel-Whitney class $w_1(E) \in \check{H}^1(B; \mathbb{Z}_2)$ in Čech cohomology with $\mathbb{Z}_2$-coefficients vanishes. For more discussion of this, see Remark 32.6 in the lecture notes.

Problem 3

The *complex projective n-space* is defined as the set of all complex lines through the origin in $\mathbb{C}^{n+1}$: more precisely, $\mathbb{CP}^n = (\mathbb{C}^{n+1} \backslash \{0\}) / \sim$, where the equivalence relation $v \sim w$ for $v, w \in \mathbb{C}^{n+1} \backslash \{0\}$ means $v = \lambda w$ for some $\lambda \in \mathbb{C}$. It is conventional to denote the equivalence class represented by $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \backslash \{0\}$ by $[z_0 : \ldots : z_n] \in \mathbb{CP}^n$. For $j = 0, \ldots, n$, define the open subset $\mathcal{U}_j := \{[z_0 : \ldots : z_n] \in \mathbb{CP}^n \mid z_j \neq 0\}$ and a map $\varphi_j : \mathbb{C}^n \to \mathbb{CP}^n$ by

$$\varphi_j(w_1, \ldots, w_n) := [w_1 : \ldots : w_j : 1 : w_{j+1} : \ldots : w_n].$$

(a) Show that for each $j = 0, \ldots, n$, φ_j is an injective map onto $\mathcal{U}_j$, thus its inverse defines a chart.

[1] If B is a smooth manifold, then open coverings with this property can always be found, e.g. by choosing a Riemannian metric and taking each $\mathcal{U}_\alpha \subset B$ to be a small geodesically convex ball around a point (cf. Problem 6).

(b) Show that the charts $\varphi_j^{-1} : \mathcal{U}_j \to \mathbb{C}^n$ for $j = 0, \ldots, n$ define an atlas on $\mathbb{CP}^n$ such that all transition maps are holomorphic. (In other words, they define a *complex structure* on $\mathbb{CP}^n$, making it an *n-dimensional complex manifold*, as well as a $2n$-dimensional smooth manifold.)

(c) Convince yourself that, as a smooth 2-manifold, $\mathbb{CP}^1$ is diffeomorphic to S^2.
Hint: It might help to first identify $\mathbb{CP}^1$ with the "extended" complex plane $\mathbb{C} \cup \{\infty\}$.

(d) For an open subset $\mathcal{O} \subset \mathbb{CP}^n$, a function $f : \mathcal{O} \to \mathbb{C}$ is called *holomorphic* if the functions $f \circ \varphi_j$ defined on the open sets $\varphi_j^{-1}(\mathcal{O}) \subset \mathbb{C}^n$ are holomorphic for each $j = 0, \ldots, n$, i.e. f "looks holomorphic" when expressed in any holomorphic chart. Find an example of an open subset $\mathcal{O} \subset \mathbb{CP}^n$ on which there exists an infinite-dimensional space of holomorphic functions, but show that the space of holomorphic functions defined *globally* on $\mathbb{CP}^n$ is finite dimensional. (What are they?)

Problem 4

For a smooth vector bundle $E \to M$ over the field $\mathbb{F}$, the *dual bundle* $E^* \to M$ is defined to have fibers $E_p^* := \mathrm{Hom}(E_p, \mathbb{F})$ for $p \in M$, and any connection ∇ on $E \to M$ naturally induces a connection on $E^* \to M$ that is uniquely determined by the Leibniz rule[2]

$$\mathcal{L}_X \left(\lambda(\eta)\right) = (\nabla_X \lambda)(\eta) + \lambda(\nabla_X \eta)$$

for $X \in \mathfrak{X}(M) = \Gamma(TM)$, $\eta \in \Gamma(E)$ and $\lambda \in \Gamma(E^*)$. Given a chart $(x^1, \ldots, x^n)$ for M and a frame $e_1, \ldots, e_m$ for E over some open set $\mathcal{U} \subset M$, let $e_*^1, \ldots, e_*^m$ denote the dual frame for E^* over the same set, determined by the condition that $e_*^a(e_b)$ is the Kronecker delta δ_b^a for each a, b. Sections $\lambda \in \Gamma(E^*)$ can now be written on $\mathcal{U}$ in the form $\lambda = \lambda_a e_*^a$ for suitable component functions $\lambda_a : \mathcal{U} \to \mathbb{F}$. Show that the Christoffel symbols $\Gamma_{ib}^a = (\nabla_i e_b)^a$ of the connection ∇ on E determine the induced connection on E^* over $\mathcal{U}$ according to the formulas

$$(\nabla_i e_*^b)_a = -\Gamma_{ia}^b \qquad \text{and} \qquad (\nabla_i \lambda)_a = \partial_i \lambda_a - \Gamma_{ia}^b \lambda_b.$$

Problem 5

Suppose $g = \langle\ ,\ \rangle \in \Gamma(E_2^0)$ is a smooth bundle metric on a real vector bundle $E \to M$. Show that the following three conditions for the "compatibility" of g with a connection ∇ on E are equivalent to each other:

(i) $\mathcal{L}_X \langle \eta, \xi \rangle = \langle \nabla_X \eta, \xi \rangle + \langle \eta, \nabla_X \xi \rangle$ for all $\eta, \xi \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$;

(ii) $\nabla g \equiv 0$ for the induced connection[3] on $E_2^0 \to M$;

(iii) The parallel transport maps $P_\gamma^t : E_{\gamma(0)} \to E_{\gamma(t)}$ along any path $\gamma(t) \in M$ satisfy $\langle P_\gamma^t(v), P_\gamma^t(w) \rangle = \langle v, w \rangle$ for all $v, w \in E_{\gamma(0)}$.

Problem 6

Given a manifold M with a chart $M \overset{\text{open}}{\supset} \mathcal{U} \xrightarrow{(x^1,\ldots,x^n)} \mathbb{R}^n$ and affine connection ∇, suppose $\gamma(t) \in \mathcal{U}$ is a nonconstant geodesic segment with image in $\mathcal{U}$, write $\gamma^i := x^i \circ \gamma$ for $i = 1, \ldots, n$ and let $\rho(t) := [\gamma^1(t)]^2 + \ldots + [\gamma^n(t)]^2$. Prove: there exists an $\epsilon > 0$ such that $\rho''(t) > 0$ whenever $\rho(t) < \epsilon$. What can you conclude about geodesics in small coordinate balls about a point?
Hint: Using the geodesic equation, derive a formula for $\rho''(t)$ involving no second derivatives of the γ^i. Then prove and make use of the estimate $\left|\sum_{i,j} \dot\gamma^i \dot\gamma^j\right| \leqslant n^2 \sum_k (\dot\gamma^k)^2$.

[2]Here $\mathcal{L}_X : C^\infty(M) \to C^\infty(M)$ denotes the derivation naturally associated to any vector field $X \in \mathfrak{X}(M)$, i.e. $(\mathcal{L}_X f)(p) := df(X(p))$.

[3]The induced connection on E_2^0 is uniquely determined by the Leibniz rule $\mathcal{L}_X \left(g(\eta, \xi)\right) = (\nabla_X g)(\eta, \xi) + g(\nabla_X \eta, \xi) + g(\eta, \nabla_X \xi)$ for all $g \in \Gamma(E_2^0)$, $\eta, \xi \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$; see §33.2 in the lecture notes.