Humboldt-Universität zu Berlin
Institut für Mathematik
C. Wendl, M. Kegel

Differentialgeometrie II

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Problem Set 10

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Problem 1

Recall that on an oriented manifold M with a fixed volume form $d\text{vol} \in \Omega^n(M)$, the *divergence* of a vector field $X \in \mathfrak{X}(M)$ is defined as the unique function $\text{div}(X) : M \to \mathbb{R}$ such that $\mathcal{L}_X(d\text{vol}) = \text{div}(X) \cdot d\text{vol}$.

(a) Prove that if $d\text{vol}$ is the canonical volume form on an oriented Riemannian manifold (M, g) and ∇ is the Levi-Cività connection, the divergence of a vector field X is given by $\text{div}(X) = \text{tr}(\nabla X)$.

(b) Prove that for $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, the divergence satisfies the Leibniz rule

$$\text{div}(fX) = f\ \text{div}(X) + df(X).$$

Problem 2

Prove that on an n-dimensional oriented Riemannian manifold (M, g), the Hodge star operator $* : \Lambda^k T^*M \to \Lambda^{n-k}T^*M$ satisfies $*^2 = (-1)^{k(n-k)}$ for each $k = 0, \ldots, n$.

Problem 3

Suppose $E, F \to M$ are vector bundles equipped with positive bundle metrics $\langle\ ,\ \rangle_E$ and $\langle\ ,\ \rangle_F$ respectively, and M is equipped with a volume form $d\text{vol} \in \Omega^n(M)$.

(a) Show that every linear differential operator $D : \Gamma(E) \to \Gamma(F)$ of order $m \in \mathbb{N}$, has a unique *formal adjoint*, meaning a differential operator $D^* : \Gamma(F) \to \Gamma(E)$ such that

$$\int_M \langle \xi, D\eta \rangle_F \, d\text{vol} = \int_M \langle D^*\xi, \eta \rangle_E \, d\text{vol}$$

for all $\xi \in \Gamma(F)$ and $\eta \in \Gamma(E)$ with compact support in $M \backslash \partial M$. Moreover, D^* has order m.

(b) Show that, in general, the formal adjoint D^* depends on the choice of volume form $d\text{vol} \in \Omega^n(M)$, but its highest-order term does not, i.e. if D_1^* and D_2^* are formal adjoints of D defined via two choices of volume form for M, then $D_1^* - D_2^* : \Gamma(F) \to \Gamma(E)$ is a differential operator of order strictly less than m.

Problem 4

Here is a quick review of a definition from the lecture: for vector bundles $E, F \to M$ and a linear differential operator $D : \Gamma(E) \to \Gamma(F)$ of order $m \in \mathbb{N}$, the *principal symbol* of D is the unique smooth (but not necessarily linear) fiber-preserving map $\sigma_D : T^*M \to \text{Hom}(E, F)$ such that for all $p \in M$, $\lambda \in T_p^*M$, $v \in E_p$, $\eta \in \Gamma(E)$ with $\eta(p) = v$ and $f \in C^\infty(M)$ with $f(p) = 0$ and $d_pf = \lambda$,

$$\sigma_D(\lambda)v = \frac{1}{m!} D(f^m\eta)(p) \in F_p.$$

(a) Prove: for $D_1 : \Gamma(E) \to \Gamma(F)$ and $D_2 : \Gamma(F) \to \Gamma(G)$ differential operators of orders m_1 and m_2 respectively, $D_2 \circ D_1 : \Gamma(E) \to \Gamma(G)$ is a differential operator of order at most $m_1 + m_2$, and if the order is exactly $m_1 + m_2$, its principal symbol is

$$\sigma_{D_2 D_1}(\lambda) = \sigma_{D_2}(\lambda)\sigma_{D_1}(\lambda) \neq 0 \in \text{Hom}(E_p, G_p) \tag{1}$$

for $\lambda \in T_p^*M$, $p \in M$. (Can you think of an example where the order is $< m_1 + m_2$?)

(b) Assuming bundle metrics on E, F and a volume form on M have been chosen, show that the principal symbols of an operator $D : \Gamma(E) \to \Gamma(F)$ of order $m \in \mathbb{N}$ and its formal adjoint $D^* : \Gamma(F) \to \Gamma(E)$ are related by

$$\sigma_{D^*}(\lambda) = (-1)^m \sigma_D(\lambda)^\dagger \in \mathrm{Hom}(F_p, E_p)$$

for $\lambda \in T_p^* M$ and $p \in M$, where $\dagger$ denotes the adjoint for linear maps $E_p \to F_p$ with respect to the bundle metrics on E and F.

Problem 5
As shown in lecture (with some details furnished by Problem 4 above), the Laplace-Beltrami operator $\Delta : \Omega^*(M) \to \Omega^*(M)$ on an oriented Riemannian manifold (M, g) has principal symbol $\sigma_\Delta : T^*M \to \mathrm{End}(\Lambda^* T^* M)$ given by $\sigma_\Delta(\lambda)\omega = -|\lambda|^2 \omega$. Deduce from this the following local coordinate expression for the second-order term in Δ: choosing a chart $(x^1, \dots, x^n)$ over $\mathcal{U} \subset M$ and writing $g = g_{ij}\, dx^i\, dx^j$ and $\omega = \omega_{i_1 \dots i_k}\, dx^{i_1} \wedge \dots \wedge dx^{i_k} \in \Omega^k(M)$ on $\mathcal{U}$,

$$(\Delta\omega)_{i_1 \dots i_k} = -g^{ab} \partial_a \partial_b \omega_{i_1 \dots i_k} + \dots,$$

where the dots indicate further terms that involve only zeroth and first derivatives of the components of ω.

Problem 6

(a) A connection on a vector bundle $E \to M$ can be regarded as a first-order linear differential operator $\nabla : \Gamma(E) \to \Gamma(\mathrm{Hom}(TM, E))$. What is the principal symbol σ_∇? Is ∇ ever elliptic?

(b) Assume M is a complex n-manifold, so its tangent spaces are naturally complex vector spaces. We can associate to any complex vector bundle $E \to M$ another complex vector bundle $F := \overline{\mathrm{Hom}}(TM, E)$ whose fiber over a point $p \in M$ is the space of complex-antilinear maps $T_p M \to E_p$. A *Cauchy-Riemann type* operator on $E \to M$ is a first-order linear differential operator $D : \Gamma(E) \to \Gamma(F)$ that satisfies the Leibniz rule

$$D(f\eta) = f\, D\eta + \bar{\partial} f(\cdot)\eta \qquad \text{for all} \qquad \eta \in \Gamma(E),\ f \in C^\infty(M, \mathbb{C}),$$

where we define $\bar{\partial} f \in \Omega^1(M, \mathbb{C})$ by $\bar{\partial} f(X) := df(X) + i\, df(iX)$. Show that all Cauchy-Riemann type operators on $E \to M$ have the same principal symbol. What is it? Are they ever elliptic?

Problem 7
Write down the principal symbol of the Dirac operator $D : \Gamma(E) \to \Gamma(E)$ on a spinor bundle $E \to M$ over a pseudo-Riemannian manifold (M, g) with a spin structure. Under what conditions is D elliptic?

Problem 8
Prove (without appealing to de Rham's theorem or other topics not covered in this course) that on a closed oriented and connected n-manifold M, an n-form $\omega \in \Omega^n(M)$ is exact if and only if $\int_M \omega = 0$.

Problem 9
It was proved in lecture that for any elliptic differential operator $D : C^\infty(\mathbb{R}^n, \mathbb{F}^k) \to C^\infty(\mathbb{R}^n, \mathbb{F}^\ell)$ of order $m \in \mathbb{N}$ with constant coefficients, any solution to the equation $D\eta = 0$ that belongs to the Sobolev space $H^m(\mathbb{R}^n)$ must be smooth. Can you find a weaker condition than ellipticity that still implies this result?