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Differentialgeometrie II

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Problem Set 3

To be discussed: 18.05.2022

Problem 1

Prove the following formulas stated in lecture for the Ricci curvature $\mathrm{Ric} \in \Gamma(T_2^0 M)$ and scalar curvature $\mathrm{Scal} \in C^\infty(M)$ of a pseudo-Riemannian manifold (M, g): for any orthonormal[1] basis $e_1, \dots, e_n \in T_pM$ at a point $p \in M$ and any $X, Y \in T_pM$,

$$\mathrm{Ric}(X, Y) = \sum_{j=1}^{n} \langle e_j, e_j \rangle \cdot \mathrm{Riem}(e_j, e_j, X, Y),$$

$$\mathrm{Scal}(p) = \sum_{i=1}^{n} \sum_{j=1}^{n} \langle e_i, e_i \rangle \cdot \langle e_j, e_j \rangle \cdot \mathrm{Riem}(e_i, e_i, e_j, e_j).$$

In the case $\dim M = 2$, deduce from this the formula $\mathrm{Scal} = 2K_G$ relating the scalar and Gaussian curvatures.

Problem 2

Show that on any pseudo-Riemannian 2-manifold (Σ, g), the Ricci curvature is determined by the Gaussian curvature and the metric via the formula $\mathrm{Ric} = K_G \cdot g$.

Problem 3

On S^2 with its standard metric, how many points is a given point p conjugate to?
Suggestion: Prove it by drawing a picture and mumbling something about $T(\exp_p)$.

Problem 4

On a pseudo-Riemannian manifold (M, g) with a geodesic segment $\gamma : [a, b] \to M$ from $\gamma(a) = p$ to $\gamma(b) = q$, prove that the following conditions are equivalent:

(i) p and q are not conjugate along γ;

(ii) For all $X \in T_pM$ and $Y \in T_qM$, there exists a unique Jacobi vector field $\eta \in \Gamma(\gamma^*TM)$ satisfying $\eta(a) = X$ and $\eta(b) = Y$.

Problem 5

By a corollary of the Cartan-Hadamard theorem discussed in lecture, there exists a unique geodesic in any given homotopy class of paths between two given points $p, q \in M$ in a complete Riemannian manifold (M, g) with nonpositive sectional curvature $K_S \leqslant 0$. Prove that in this situation, the geodesic in question is also strictly shorter than all other paths (excepting its own reparametrizations) from p to q in the same homotopy class.
Hint: If $\pi : \widetilde{M} \to M$ *denotes the universal cover,* $\widetilde{g} := \pi^*g$ *defines a metric on* $\widetilde{M}$ *such that paths in* (M, g) *have the same lengths as their lifts in* $(\widetilde{M}, \widetilde{g})$*, and geodesics lift to geodesics.*

Problem 6

Suppose G is a Lie group with identity element $e \in G$ and Lie algebra $\mathfrak{g} = T_eG$, and $i : G \to G$ denotes the inversion map $g \mapsto g^{-1}$. Prove:

[1] Since g might not be positive-definite, "orthonormal" in this case means $g(e_i, e_j) = \pm\delta_{ij}$, where the signs may vary depending on the signature of g.

(a) The derivative $T_e i : \mathfrak{g} \to \mathfrak{g}$ of i at e is multiplication by -1.
Suggestion: There is an easy proof using some knowledge of the exponential map, but try to find a more basic proof that does not require such knowledge.

(b) G is parallelizable, i.e. the tangent bundle $TG \to G$ is trivial.

(c) A tensor field $S \in \Gamma(T^k_\ell G)$ is left-invariant if and only if $i^* S \in \Gamma(T^k_\ell M)$ is right-invariant.

(d) $X^R = -i^* X^L$ for any $X \in \mathfrak{g}$, where $X^L, X^R \in \mathfrak{X}(G)$ denote the unique left- and right-invariant vector fields respectively that satisfy $X^L(e) = X^R(e) = X$.

(e) For $X, Y \in \mathfrak{g}$ and the corresponding right-invariant vector fields $X^R, Y^R \in \mathfrak{X}(G)$, $[X^R, Y^R](e) = -[X, Y]$.

(f) Every left- or right-invariant vector field on G has a global flow.

(g) For any $X \in \mathfrak{g}$, the flows of the corresponding left- and right-invariant vector fields $X^L, X^R \in \mathfrak{X}(G)$ are given by

$$\varphi^t_{X^L}(g) = g \exp(tX), \qquad \varphi^t_{X^R}(g) = \exp(tX) g.$$

Hint: For each $g \in G$, the left-translation diffeomorphism $L_g : G \to G$ sends flow lines of X^L to other flow lines of X^L.

Problem 7
On a Lie group G with Lie algebra $\mathfrak{g}$, prove that if $X, Y \in \mathfrak{g}$ satisfy $[X, Y] = 0$, then $\exp(X + Y) = \exp(X)\exp(Y) = \exp(Y)\exp(X)$.
Hint: Use the flows of the left-invariant vector fields $X^L, Y^L \in \mathfrak{X}(G)$ to prove $\exp(sX)\exp(tY) = \exp(tY)\exp(sX)$ for all $s, t \in \mathbb{R}$. Then show that $t \mapsto \exp(tX)\exp(tY)$ is a flow line of $(X + Y)^L$.

Problem 8
Determine the dimensions and the Lie algebras of each of the following matrix groups: $\mathrm{GL}_+(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{O}(k, \ell)$, $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$.

Problem 9
Give an explicit description of the Lie algebra of the group $\mathrm{Aff}(\mathbb{R}^n)$ of affine transformations $\mathbb{R}^n \to \mathbb{R}^n$, e.g. write down a basis and compute the Lie brackets of the basis elements.