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Differentialgeometrie II
SoSe 2022

Problem Set 4

To be discussed: 26.05.2022

Problem 1

(a) Prove that the formula $\langle \mathbf{A}, \mathbf{B} \rangle := -\operatorname{tr}(\mathbf{AB})$ defines an Ad-invariant positive inner product on the Lie algebras of $\mathrm{SO}(n)$, $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$. (For the latter two groups, you also need to show that the pairing is real valued.)

(b) Prove that the same formula defines an Ad-invariant nondegenerate (but not positive-definite) symmetric bilinear form on the Lie algebra of $\mathrm{SL}(2, \mathbb{R})$. What is its signature? *Suggestion: It isn't too hard to guess an orthonormal basis of* $\mathfrak{sl}(2, \mathbb{R})$.

Problem 2

Suppose o^L and o^R denote left- and right-invariant orientations respectively on $\mathrm{O}(2)$. Show that if these match on the identity component $\mathrm{SO}(2) \subset \mathrm{O}(2)$, then they differ on the component containing the reflection $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, thus $\mathrm{O}(2)$ has no bi-invariant orientation.

Problem 3

Consider the 2-dimensional Lie group $\mathrm{Aff}_+(\mathbb{R}) \subset \mathrm{Diff}(\mathbb{R})$ consisting of affine transformations of the form $t \mapsto at + b$ with constants $a > 0$ and $b \in \mathbb{R}$. There is a global chart (x, y) identifying $\mathrm{Aff}_+(\mathbb{R})$ with the upper half-plane $\{y > 0\} \subset \mathbb{R}^2$ such that a point (x, y) is identified with the transformation $t \mapsto yt + x$. The identity $\mathrm{Id} \in \mathrm{Aff}_+(\mathbb{R})$ thus has coordinates $(x, y) = (0, 1)$.

(a) Find the unique functions $f^L, f^R : \mathrm{Aff}_+(\mathbb{R}) \to (0, \infty)$ such that $f^L(\mathrm{Id}) = f^R(\mathrm{Id}) = 1$ and the volume forms $f^L\, dx \wedge dy, f^R\, dx \wedge dy \in \Omega^2(\mathrm{Aff}_+(\mathbb{R}))$ are left-invariant and right-invariant respectively.

(b) Show that the modular function $\Delta : \mathrm{Aff}_+(\mathbb{R}) \to \mathbb{R}_{>0}$ is given by $\Delta(x, y) = y$.

Problem 4

A Lie algebra $\mathfrak{g}$ is called *simple* if it contains no nontrivial proper Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $[X, Y] \in \mathfrak{h}$ for every $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$. In particular, if G is a Lie group whose Lie algebra $\mathfrak{g}$ is simple, then no nontrivial proper subspace of $\mathfrak{g}$ is invariant under the map $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ for every $X \in \mathfrak{g}$. Prove:

(a) $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ are simple. (cf. Problem 8(a))

(b) If G is a Lie group admitting a bi-invariant Riemannian metric and its Lie algebra $\mathfrak{g}$ is simple, then its bi-invariant metric is unique up to positive scaling.
Hint: If $\langle\ ,\ \rangle$ *and* $\langle\ ,\ \rangle'$ *are two* Ad*-invariant inner products on* $\mathfrak{g}$*, then* $\langle X, Y \rangle' = \langle X, AY \rangle$ *for a linear map* $A : \mathfrak{g} \to \mathfrak{g}$ *that is symmetric with respect to* $\langle\ ,\ \rangle$ *and commutes with* Ad_g *for every* $g \in G$*. Deduce from the latter that for every* $X \in \mathfrak{g}$*,* $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ *preserves each eigenspace of* A*, and conclude that there can only be one eigenspace.*

(c) The group $\mathrm{SO}(3) \times \mathrm{SO}(3)$ admits two bi-invariant Riemannian metrics that are not scalar multiples of each other.

Problem 5
For a Lie group G, prove:

(a) If G is unimodular, then any Haar measure satisfies $\int_G f(g)\,dg = \int_G f(g^{-1})\,dg$ for compactly-supported smooth functions f.

(b) For a left-invariant pseudo-Riemannian metric $\langle\ ,\ \rangle$, the inversion map $G \to G : g \mapsto g^{-1}$ is an isometry if and only if $\langle\ ,\ \rangle$ is bi-invariant.

Problem 6
On a Lie group G with bi-invariant Riemannian metric $\langle\ ,\ \rangle$, prove:

(a) The Riemann tensor satisfies $R(X,Y)Z = \frac{1}{4}[Z,[X,Y]]$ for any left-invariant vector fields $X, Y, Z \in \mathfrak{X}(G)$.

(b) If $P \subset T_gG$ is the plane spanned by the values of two orthonormal left-invariant vector fields X, Y at $g \in G$, then the sectional curvature satisfies $K_S(P) = \frac{1}{4}\left|[X,Y]\right|^2$.

Problem 7
Prove: If $\Phi : G \to H$ is a Lie group homomorphism between connected Lie groups such that $\Phi_* : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra isomorphism, then Φ is a covering map.
Hint: For inspiration, see the proof of Lemma 36.19 in the notes.

Problem 8
In the following, it may help to recall the Lie algebra isomorphism $(\mathbb{R}^3, \times) \to \mathfrak{so}(3)$ discussed in lecture. Prove:

(a) Every nontrivial proper Lie subalgebra of $\mathfrak{so}(3)$ is 1-dimensional.

(b) Every nontrivial Lie algebra homomorphism $\mathfrak{so}(3) \to \mathfrak{so}(3)$ is an isomorphism.
Hint: What could the dimensions of its kernel and image be?

(c) Every connected nontrivial proper Lie subgroup of $\mathrm{SO}(3)$ takes the form $\{e^{t\mathbf{A}} \in \mathrm{SO}(3) \mid t \in \mathbb{R}\}$ for some $\mathbf{A} \in \mathfrak{so}(3)$, and is thus isomorphic to $\mathrm{SO}(2) \cong S^1$.

(d) Every nontrivial Lie group homomorphism $\mathrm{SO}(3) \to \mathrm{SO}(3)$ is an isomorphism.
Hint: This is fairly easy if you apply some covering space theory, but if you prefer not to, then you can still do this if you first show that all Lie algebra isomorphisms of $(\mathbb{R}^3, \times)$ to itself are given by orthogonal transformations.

(e) There exist nontrivial Lie group homomorphisms $\mathrm{SO}(2) \to \mathrm{SO}(2)$ that are not isomorphisms.

Problem 9
Suppose G is a Lie group with an Ad-invariant positive inner product $\langle\ ,\ \rangle$ on its Lie algebra $\mathfrak{g}$.

(a) Show that for any $X, Y \in \mathfrak{g}$, $[X,Y]$ is orthogonal to both X and Y.
Hint: Use the formula $\mathrm{ad}_X Y = [X,Y]$ and the antisymmetry of $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$.

(b) Show that if $\mathfrak{g}$ is not abelian and $\dim \mathfrak{g} = 3$, then there exists a constant $\lambda > 0$ and an orthonormal basis $e_1, e_2, e_3 \in \mathfrak{g}$ satisfying $[e_1, e_2] = \lambda e_3$, $[e_2, e_3] = \lambda e_1$ and $[e_3, e_1] = \lambda e_2$. Deduce that $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(3)$.

(c) Prove that for every compact connected non-abelian Lie group G with $\dim G = 3$, there exists a Lie group homomorphism $G \to \mathrm{SO}(3)$ that is a covering map.

(d) Find an example of a connected non-abelian (but not compact!) 3-dimensional Lie group whose Lie algebra is not isomorphic to $\mathfrak{so}(3)$.