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Differentialgeometrie II
SoSe 2022

Problem Set 6

To be discussed: 8.06.2022

Problem 1
This is a reprise of a problem from last week, since the proof of the slice theorem was covered one lecture later than planned.
Assume $G \times M \to M$ is a smooth proper group action such that the isotropy subgroup $G_p \subset G$ is finite for every $p \in M$. The goal of this problem is to show, via a mild generalization of the slice theorem, that G/H is then an *orbifold* of dimension $\dim M - \dim G$, a notion generalizing the concept of a manifold. In an n-dimensional orbifold, every point has a neighborhood homeomorphic to the quotient of an open subset of $\mathbb{R}^n$ by a finite group action.

(a) Show that for each $p \in M$, the orbit $Gp \subset M$ is a smooth submanifold. Unlike the case of a free action, Gp will not generally be diffeomorphic to G. What instead?

(b) Show that p admits a G_p-invariant neighborhood $\mathcal{U} \subset M$ with a G_p-invariant Riemannian metric, i.e. each $g \in G_p$ acts on $\mathcal{U}$ via an isometry.
Hint: Start with any metric, then act on it with G_p and take an average.

(c) Construct a submanifold $\Sigma \subset M$ that satisfies $\Sigma \cap Gp = \{p\}$ and $T_p\Sigma \oplus T_p(Gp) = T_pM$ and, additionally, is invariant under the action of G_p.
Hint: Use geodesics.

(d) Show that after possibly shrinking Σ to a smaller neighborhood of p, the map $\Phi : G \times \Sigma \to M : (g, q) \mapsto gq$ can be assumed to be a local diffeomorphism satisfying $g(\Sigma) \cap \Sigma = \varnothing$ for all $g \in G \backslash G_p$.

(e) Conclude that the map $\Sigma \to M/G : p \mapsto [p]$ descends to the quotient Σ/G_p as a homeomorphism onto a neighborhood of $[p]$ in M/G.

(f) Deduce that for every $p \in M$, suffiently nearby points $q \in M$ satisfy $|G_q| \leqslant |G_p|$.

Problem 2

(a) Show that for the natural action of $\mathrm{SO}(3)$ on $S^2 \subset \mathbb{R}^3$ by linear transformations restricted to the unit sphere, the only Lie subgroup of $\mathrm{SO}(3)$ that acts transitively on S^2 is $\mathrm{SO}(3)$ itself.

(b) Show that the analogue of part (a) does not hold for $\mathrm{SO}(4)$, i.e. there exists a proper Lie subgroup $G \subset \mathrm{SO}(4)$ that acts transitively on the unit sphere $S^3 \subset \mathbb{R}^4$.
Hint: Identify $\mathbb{R}^4$ with $\mathbb{C}^2$ so that $\mathrm{GL}(2, \mathbb{C})$ becomes a subgroup of $\mathrm{GL}(4, \mathbb{R})$, and show that from this perspective, $\mathrm{SO}(4) \cap \mathrm{GL}(2, \mathbb{C}) = \mathrm{U}(2)$.

Problem 3
Given an n-dimensional vector space V over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $\mathrm{Gr}_k(V)$ denote the Grassmann manifold of k-dimensional subspaces of V. Prove that

$$E := \left\{(W, v) \in \mathrm{Gr}_k(V) \times V \;\middle|\; v \in W\right\}$$

defines a smooth subbundle of the trivial vector bundle $\mathrm{Gr}_k(V) \times V \to \mathrm{Gr}_k(V)$. (We call E the *tautological vector bundle* over $\mathrm{Gr}_k(V)$.)

Problem 4
On a real vector space V of dimension $2n$, let

$$\mathcal{J}(V) := \left\{ J \in \mathrm{End}(V) \;\middle|\; J^2 = -\mathbb{1} \right\}.$$

The elements $J \in \mathcal{J}(V)$ are called *complex structures* on V, as each one can be used to endow V with the structure of an n-dimensional complex vector space on which scalar multiplication is defined by $(a + ib)v := av + bJv$. Prove that $\mathcal{J}(V)$ is a smooth noncompact submanifold of $\mathrm{End}(V)$ with dimension $2n^2$, and that it has exactly two connected components.
Hint: Find a smooth action of $\mathrm{GL}(V)$ *on* $\mathrm{End}(V)$ *that preserves* $\mathcal{J}(V)$ *and has stabilizer at some point* $J_0 \in \mathcal{J}(V)$ *isomorphic to* $\mathrm{GL}(n, \mathbb{C})$.

Problem 5
For a smooth fiber bundle $\pi : E \to M$ and Lie group G, one can define a partial order $\prec$ on the set of all G-bundle atlases by inclusion, meaning we write $\mathcal{A}^1 \prec \mathcal{A}^2$ if $\mathcal{A}^1$ can be obtained by removing some of the trivializations and corresponding transition functions from $\mathcal{A}^2$ but otherwise changing nothing. Prove:

(a) If $\mathcal{A}^1 \prec \mathcal{A}^2$, then $\mathcal{A}^1$ and $\mathcal{A}^2$ are equivalent.

(b) If $\mathcal{A}^1$ and $\mathcal{A}^2$ are equivalent, then there exists another G-bundle atlas $\mathcal{A}^3$ such that $\mathcal{A}^1 \prec \mathcal{A}^3$ and $\mathcal{A}^1 \prec \mathcal{A}^3$.

(c) For any collection of G-bundle atlases $\{\mathcal{A}^j\}_{j \in J}$ that is totally ordered, meaning either $\mathcal{A}^j \prec \mathcal{A}^k$ or $\mathcal{A}^k \prec \mathcal{A}^j$ holds for every $j, k \in J$, there exists a G-bundle atlas $\mathcal{A}$ such that $\mathcal{A}^j \prec \mathcal{A}$ for every $j \in J$.

Comment: Students familiar with Zorn's lemma will recognize that we have just established its hypotheses. The result is that every equivalence class of G*-bundle atlases has a unique maximal representative.*

Problem 6
Here's a subject that probably should have been covered last semester but wasn't: given a smooth map $f : N \to M$ and a smooth submanifold $Q \subset M$, we say that f is *transverse* to Q and write "$f \pitchfork Q$" if for every $p \in N$ with $q := f(p) \in Q$, we have $(\operatorname{im} T_p f) + T_q Q = T_q M$.

(a) Prove that if $f \pitchfork Q$, then $\Sigma := f^{-1}(Q) \subset N$ is a smooth submanifold with $T_p\Sigma = \left\{ X \in T_p N \;\middle|\; T_p f(X) \in T_q Q \right\}$ for every $p \in \Sigma$ and $q := f(p)$. What is its dimension?
Hint: This generalizes the version of the implicit function theorem that we have often used to study level sets of smooth maps at regular values, but it also follows from that version of the theorem if you work in suitable coordinates near Q.

(b) Suppose $f : N \to M$ and $\pi : E \to M$ are two smooth maps such that π is a submersion. Prove that the map

$$f \times \pi : N \times E \to M \times M : (p, x) \mapsto (f(p), \pi(x))$$

is then transverse to the so-called *diagonal* submanifold $\Delta := \{(p, p) \mid p \in M\} \subset M \times M$, and thus $\Sigma := (f \times \pi)^{-1}(\Delta)$ is a submanifold of $N \times E$. Prove moreover that the map $\Sigma \to N : (p, x) \mapsto p$ is a submersion.

Comment: In the situation we are interested in, $\pi : E \to M$ *is a smooth fiber bundle, and the submanifold* Σ *obtained in part (b) is then the most natural model for the total space of the pullback bundle* $f^*E \to N$.