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Differentialgeometrie II
SoSe 2022

Take-Home Midterm

Due: Wednesday, 6.07.2022, 9:15am (100pts total)

Instructions

The purpose of this assignment is three-fold:

- In the absence of regular problem sets for the next two weeks, it deals in part with current material from the lectures.
- It gives the instructors a chance to gauge your understanding more directly than usual, and give feedback.
- It provides an opportunity to improve your final grade in the course.

To receive feedback and/or credit, you must submit your written solutions before the beginning of the **Übung** on **Wednesday, July 6 at 9:15**. Submissions can be on paper or electronic; in the latter case, you should e-mail them to `wendl@math.hu-berlin.de` by the deadline. The solutions will be discussed in the Übung on the due date.

You are free to use any resources at your disposal and to discuss the problems with your comrades, but **you must write up your solutions alone**. Solutions may be written up in German or English, this is up to you.

A score of 60 points or better will boost your final exam grade according to the formula that was indicated in the course syllabus. The number of points assigned to each part of each problem is meant to be approximately proportional to its importance and/or difficulty.

If a problem asks you to prove something, then unless it says otherwise, a **complete argument** is typically expected, not just a sketch of the idea. Partial credit may sometimes be given for incomplete arguments if you can demonstrate that you have the right idea, but for this it is important to write as clearly as possible. Less complete arguments can sometimes be sufficient, e.g. if you need to choose a smooth cutoff function with particular properties and can justify its existence with a convincing picture instead of an explicit formula (use your own judgement). Unless stated otherwise, you are free to make use of all results that have appeared in the lecture notes or in problem sets, without reproving them. When using a result from a problem set or the lecture notes, say explicitly which one.

If you get stuck on one part of a problem, it may often still be possible to move on and do the next part. You are free to ask for clarification or hints via e-mail/moodle or in office hours or Übungen; of course we reserve the right not to answer such questions.

Problem 1 [30 pts]
In this problem, (M, g) is a Riemannian manifold of dimension $n \in \mathbb{N}$, and we denote its isometry group by

$$\mathrm{Isom}(M, g) \subset \mathrm{Diff}(M).$$

It will be useful to recall that isometries $\varphi \in \mathrm{Isom}(M)$ preserve all objects whose definitions depend only on the metric, e.g. the condition $\varphi^* g = g$ implies $\varphi^* R = R$, $\varphi^* \mathrm{Ric} = \mathrm{Ric}$ and $\mathrm{Scal} \circ \varphi = \mathrm{Scal}$ for the Riemann, Ricci and scalar curvatures respectively, and a path $\gamma : (a, b) \to M$ is a geodesic if and only if $\varphi \circ \gamma : (a, b) \to M$ is a geodesic.

We call (M, g) **homogeneous** if for every pair of points $p, q \in M$ there exists an isometry $\varphi \in \mathrm{Isom}(M, g)$ with $\varphi(p) = q$. We say moreover that (M, g) is **isotropic** at a point $p \in M$ if for every pair of unit vectors $X, Y \in T_p M$, there exists an isometry $\varphi \in \mathrm{Isom}(M, g)$ such that $\varphi(p) = p$ and $T\varphi(X) = Y$.

(a) [10 pts] Assuming (M, g) is connected, show that it is homogeneous and isotropic at one point if and only if it is isotropic at every point. (One says in this case that (M, g) is “homogeneous and isotropic”.)

We call g an **Einstein metric** on M if its Ricci tensor $\mathrm{Ric} \in \Gamma(T_2^0 M)$ is a scalar multiple of the metric g at every point, that is, $\mathrm{Ric} = f \cdot g$ for some function $f : M \to \mathbb{R}$. Problem Set 3 #2 implies for instance that *every* Riemannian metric on a 2-manifold is Einstein. We will show later that when $\dim M \geqslant 3$, “most” metrics are not Einstein, because all Einstein metrics have constant scalar curvature. (I mention this just to give some context, but for the present problem, you do not need to know it.)

(b) [5 pts] Show that if g is an Einstein metric, then the function $f : M \to \mathbb{R}$ in the definition above is a positive constant multiple of the scalar curvature $\mathrm{Scal} : M \to \mathbb{R}$, and the constant depends only on the dimension of M. (What is it?)

(c) [15 pts] Show that if (M, g) is homogeneous and isotropic, then g is Einstein.
Hint: For each $p \in M$, $\mathrm{Ric}_p = g(R_p^\sharp \cdot, \cdot)$ for a symmetric linear map $R_p^\sharp : T_p M \to T_p M$. What can you say about the eigenspaces of $R_p^\sharp$ and the action of the stabilizer $G_p := \{\varphi \in \mathrm{Isom}(M, g) \mid \varphi(p) = p\}$ on $T_p M$?

Problem 2 [20 pts]
For a principal G-bundle $\pi : E \to M$ with connection $A \in \Omega^1(E, \mathfrak{g})$ and curvature $F_A \in \Omega^2(E, \mathfrak{g})$, the second Bianchi identity says

$$dF_A = [F_A, A],$$

and it is equivalent to the statement that $d_A F_A = 0$ for the operator $d_A = d + A \wedge (\cdot)$ on the space of horizontal Ad-equivariant forms $\Omega^*_{\mathrm{Ad}}(E, \mathfrak{g})$. Under the natural isomorphism $\Omega^*_{\mathrm{Ad}}(E, \mathfrak{g}) \cong \Omega^*(M, \mathrm{Ad}(E))$, the identity thus becomes

$$d_\nabla \Omega_A = 0$$

for the bundle-valued curvature 2-form $\Omega_A \in \Omega^2(M, \mathrm{Ad}(E))$ equivalent to F_A.

(a) [16 pts] Derive from this the so-called **differential Bianchi identity**, which states that for the Levi-Cività connection on a Riemannian manifold (M, g), the covariant derivative of the Riemann tensor $R \in \Gamma(T_3^1 M)$ satisfies

$$(\nabla_X R)(Y, Z) + (\nabla_Y R)(Z, X) + (\nabla_Z R)(X, Y) = 0 \qquad \text{for all } X, Y, Z \in \mathfrak{X}(M).$$

To clarify the notation: for $X \in \mathfrak{X}(M)$ we are viewing $\nabla_X R \in \Gamma(T^1_3 M)$ as a multilinear bundle map $TM \oplus TM \oplus TM \to TM : (Y, Z, V) \mapsto (\nabla_X R)(Y, Z)V$, just as with R.

(b) [4 pts] Under what assumptions on an affine connection ∇ is the differential Bianchi identity valid? Does it really need to be the Levi-Cività connection of a metric?

Hint: Given $V \in \mathfrak{X}(M) = \Omega^0(M, TM)$, try applying the operator d_∇ to both sides of the formula $R(\cdot, \cdot)V = \Omega_A \wedge V \in \Omega^2(M, TM)$ proved in lecture. For the ensuing calculation, it may help to rewrite the lecture notes' formula (45.3) for the covariant exterior derivative of a bundle-valued 2-form $\omega \in \Omega^2(M, E)$ as

$$d_\nabla \omega(X, Y, Z) = \nabla_X \big(\omega(Y, Z)\big) - \omega([X, Y], Z) + \text{cyclic},$$

where the word "cyclic" means that additional (in this case four) terms appear, obtained from the written terms via all possible cyclic permutations of the triple (X, Y, Z). Similarly, for a bilinear bundle map $\mu : E_1 \oplus E_2 \to F$ and forms $\omega \in \Omega^2(M, E_1)$, $\lambda \in \Omega^1(M, E_2)$, the formula (45.1) in the notes becomes

$$\mu(\omega, \lambda)(X, Y, Z) = \mu(\omega(X, Y), \lambda(Z)) + \text{cyclic}.$$

One last piece of advice: when brackets appear, use the torsion tensor to get rid of them.

Problem 3 [25 pts]
The goal of this problem is to exhibit an example of a Lie group G that is not isomorphic to any matrix group; in fact it does not even admit an injective Lie group homomorphism into $\mathrm{GL}(m, \mathbb{R})$ for any m. You are free to use as black boxes the following three results, the first of which is standard in elementary algebraic topology, and proofs of the other two are sketched in §39.3 of the notes.

- The sphere S^n is simply connected for every $n \geqslant 2$, but S^1 is not.
- If G, H are two Lie groups and G is simply connected, then every Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ is the derivative at $e \in G$ of a unique Lie group homomorphism $\Phi : G \to H$.
- For every connected Lie group G, there is a simply connected Lie group $\widetilde{G}$ admitting a Lie group homomorphism $\Phi : \widetilde{G} \to G$ whose derivative at the identity is an isomorphism $\Phi_* : \widetilde{\mathfrak{g}} \to \mathfrak{g}$. (The latter implies via Problem Set 4 #7 that Φ is a covering map, hence it is surjective, and is injective if and only if G is also simply connected. We call $\widetilde{G}$ the **universal cover** of G.)

Prove:

(a) [7 pts] $\mathrm{SL}(2, \mathbb{C})$ is simply connected, but $\mathrm{SL}(2, \mathbb{R})$ is not.
Hint: Are you familiar with polar decomposition of matrices? Show that any continuous loop in $\mathrm{SL}(2, \mathbb{C})$ or $\mathrm{SL}(2, \mathbb{R})$ can be deformed continuously to a loop in $\mathrm{SU}(2)$ or $\mathrm{SO}(2)$ respectively. This is analogous to the fact that loops in $\mathbb{R}^n \backslash \{0\}$ can be deformed to loops in S^{n-1}.

(b) [8 pts] For every $m \in \mathbb{N}$, every Lie algebra homomorphism $\phi : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{gl}(m, \mathbb{R})$ is the derivative at $\mathbb{1}$ of a unique Lie group homomorphism $\Phi : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{GL}(m, \mathbb{R})$.
Hint: What relation is there between $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{sl}(2, \mathbb{C})$?

(c) [10 pts] For the universal cover $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ of $\mathrm{SL}(2,\mathbb{R})$, every Lie group homomorphism $\Phi : \widetilde{\mathrm{SL}(2,\mathbb{R})} \to \mathrm{GL}(m,\mathbb{R})$ for $m \in \mathbb{N}$ is the composition of the covering map $\widetilde{\mathrm{SL}(2,\mathbb{R})} \to \mathrm{SL}(2,\mathbb{R})$ with a Lie group homomorphism $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{GL}(m,\mathbb{R})$, so in particular, Φ can never be injective.

Problem 4 [25 pts]
For a principal U(1)-bundle $\pi : E \to M$ with connection $A \in \Omega^1(E,\mathfrak{g})$, we saw in lecture that the bundle-valued curvature 2-form $\Omega_A \in \Omega^2(M, \mathrm{Ad}(E))$ can be regarded as a pure imaginary-valued 2-form

$$\Omega_A \in \Omega^2(M, \mathfrak{u}(1)) = \Omega^2(M, i\mathbb{R}),$$

which is closed due to the second Bianchi identity, and the first Chern class is then defined as $c_1(E) = [(i/2\pi)\Omega_A] \in H^2_{\mathrm{dR}}(M)$.

(a) [10 pts] Show that for a closed 2-form $\omega \in \Omega^2(M)$, if $[\omega] = c_1(E)$, then the bundle $\pi : E \to M$ admits a principal connection whose curvature 2-form (regarded as an imaginary-valued 2-form on M) is $\frac{2\pi}{i}\omega$.

Regarding S^{2n+1} as the unit sphere in $\mathbb{C}^{n+1}$, the group $\mathrm{U}(1) \subset \mathrm{GL}(1,\mathbb{C}) = \mathbb{C}^*$ acts on S^{2n+1} via scalar multiplication, thus defining a principal U(1)-bundle

$$\pi : S^{2n+1} \to \mathbb{CP}^n : (z_0, \dots, z_n) \mapsto [z_0 : \dots : z_n]$$

that can also be viewed as the orthonormal frame bundle of the tautological line bundle $E \to \mathbb{CP}^n$ with its canonical bundle metric (cf. Problem Set 8 #3). Writing $\langle z, w\rangle := \sum_{j=0}^n \bar{z}^j w^j$ for the standard Hermitian inner product on $\mathbb{C}^{n+1}$, we can define a 1-form $\lambda \in \Omega^1(\mathbb{C}^{n+1})$ and 2-form $\omega \in \Omega^2(\mathbb{C}^{n+1})$ by

$$\lambda_z(X) := \mathrm{Re}\langle iz, X\rangle, \qquad \omega_z(X,Y) := \mathrm{Re}\langle iX, Y\rangle$$

for $z \in \mathbb{C}^{n+1}$ and $X, Y \in T_z\mathbb{C}^{n+1} = \mathbb{C}^{n+1}$. Notice that ω satisfies $\omega(X, iX) > 0$ whenever $X \neq 0$.

(b) [5 pts] Show that $A := i\lambda|_{TS^{2n+1}} \in \Omega^1(S^{2n+1}, \mathfrak{u}(1))$ is a connection 1-form for the principal bundle $\pi : S^{2n+1} \to \mathbb{CP}^n$, and the resulting horizontal subspace $H_z S^{2n+1} \subset T_z S^{2n+1}$ for each $z \in S^{2n+1}$ is a complex subspace of $\mathbb{C}^{n+1}$.

(c) [5 pts] Show that for the connection in part (b), the curvature 2-form is $F_A = 2i\omega|_{TS^{2n+1}}$.

(d) [5 pts] Show that the first Chern class of the bundle $\pi : S^{2n+1} \to \mathbb{CP}^n$ can be represented by a closed 2-form $\alpha \in \Omega^2(\mathbb{CP}^n)$ that satisfies $\int_\Sigma \alpha \neq 0$ for every closed complex 1-dimensional submanifold $\Sigma \subset \mathbb{CP}^n$ (which is also a canonically oriented real 2-dimensional submanifold). Conclude that the first Chern class is nonzero. You may use without proof the following fact: the quotient projection $\Pi : \mathbb{C}^{n+1}\backslash\{0\} \to (\mathbb{C}^{n+1}\backslash\{0\})/\mathbb{C}^* = \mathbb{CP}^n$ is a holomorphic map between complex manifolds, implying in particular that $T_z\Pi : \mathbb{C}^{n+1} = T_z(\mathbb{C}^{n+1}\backslash\{0\}) \to T_{[z]}\mathbb{CP}^n$ is complex linear for every $z \in \mathbb{C}^{n+1}\backslash\{0\}$.