# Short synopsis

Catastrophe theory is a slightly sensationalistic term for a branch of pure mathematics that can be used to model sudden or violent (i.e. discontinuous) changes in systems that depend smoothly on external parameters. The subject earned a mildly dubious reputation during the 1960's and 70's, due to a certain amount of overzealous hype about its applications to the natural sciences and humanities.[1] But as a purely mathematical discipline, catastrophe theory is an elegant synthesis of differential calculus with commutative algebra, one that furnishes answers to many important questions arising in the study of dynamical systems, differential geometry, topology, and other areas of both pure and applied mathematics. Catastrophe theory is also a special case of—and an accessible entry point into—the larger subjects of *singularity theory* and *bifurcation theory*, which study the qualitative structure of smooth maps (and their dependence on extra parameters) near points at which the usual hypotheses of the inverse and implicit function theorems fail.

In this seminar, we will mostly *not* discuss applications, but focus instead on the mathematical underpinnings of catastrophe and singularity theory. Our first major goal will be to understand a famous result of René Thom, which classifies the qualitative structure of all possible catastrophes for systems depending on at most four parameters: in essence, every such system that can arise in practice matches one of seven explicit local models, known as the *seven elementary catastrophes*.[2] After this, we will have some time to discuss how the ideas behind that theorem generalize into the wider contexts of singularity and bifurcation theory.

The essential **prerequisites** for this seminar are covered by the HU's standard sequence of required Bachelor-level courses, namely *Analysis I–III*, *Lineare Algebra I–II* and *Algebra und Funktionentheorie*. More precisely, participants will need to have a solid understanding of differential calculus for maps $\mathbb{R}^n \to \mathbb{R}^m$, the basic existence/uniqueness theory for ordinary differential equations (as covered in *Analysis III*), and some basic knowledge of rings, ideals and modules. Further knowledge of commutative algebra is not required, as the necessary tools (e.g. Nakayama's lemma) will be introduced in the course of the seminar. Knowledge of the theory of smooth manifolds is not necessary, but may occasionally be helpful; we will at least frequently mention the notion of *submanifolds* of $\mathbb{R}^n$ and their tangent spaces, as arise for instance from standard applications of the implicit function theorem.

# Some basic notions

For a more concrete idea of what this subject is about, here are a few of the fundamental definitions, followed by an illustrative example.

## Singularities

Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth (i.e. infinitely differentiable) map. After adding a constant, we may as well assume that $f(0) = 0$, and let us denote its derivative at that point by $Df(0) : \mathbb{R}^n \to \mathbb{R}^m$, regarded as a linear map. According to the inverse function theorem, if $Df(0)$ is invertible (only possible of course if $m = n$), then $f$ maps some neighborhood of 0 bijectively to some neighborhood of 0, and its inverse on the latter neighborhood is also smooth. One can similarly use the implicit function theorem to show the following:

- If $Df(0)$ is surjective (only possible if $n \geq m$), then the image of $f$ on a neighborhood of $0 \in \mathbb{R}^n$

[1] For an entertaining underhanded rant on this topic, see [AGZV12, pp. 17–18].

[2] If you happen to have recently formed a band with six other people, but have not yet thought up a good name for it, I suggest you consider *The Seven Elementary Catastrophes*. You're welcome.

contains a neighborhood of $0 \in \mathbb{R}^m$, and moreover, there is a smooth bijective correspondence between a neighborhood of 0 in the preimage $f^{-1}(0) \subset \mathbb{R}^n$ and a neighborhood of 0 in the $(n-m)$-dimensional vector space $\ker Df(0) \subset \mathbb{R}^n$. In particular, this makes $f^{-1}(0)$ into a smooth $(n-m)$-dimensional **submanifold** of $\mathbb{R}^n$ near the origin.

- If $Df(0)$ is injective (only possible if $n \le m$), then $f$ is injective on some neighborhood of 0 and the image of that neighborhood is a smooth $n$-dimensional submanifold of $\mathbb{R}^m$.

If none of these hypotheses hold, then the implicit function theorem does not provide any straightforward qualitative description of either the preimage $f^{-1}(0)$ or the image of $f$ near the origin, and we say in this case that $f$ has a **singularity** at 0. One of the major goals of singularity theory is to do in this situation what the implicit function theorem cannot: to describe the local structure of $\operatorname{im} f$ and/or $f^{-1}(0)$ in a neighborhood of the singularity. One of the deep insights of the subject is that such descriptions can typically be deduced from algebraic information determined by the Taylor coefficients of $f$ at 0, and if one understands the algebra well enough, then it is often even possible to write down a precise list of all possible local models that can describe $f$ up to a change of coordinates.

## Potentials and equilibria

Catastrophe theory studies the singularities of a specific class of smooth maps $f : \mathbb{R}^n \to \mathbb{R}^m$ with $n = m$, namely vector fields $f = \nabla F = (\partial_1 F, \dots, \partial_n F) : \mathbb{R}^n \to \mathbb{R}^n$ that arise as gradients of smooth "potential" functions

$$F : \mathbb{R}^n \to \mathbb{R}.$$

There is some real-world motivation for this: as you may recall from elementary physics, mechanical systems that obey the conservation of energy can typically be described via their potential energy, a smooth function $F : \mathbb{R}^n \to \mathbb{R}$, where $n$ in this setting is the number of degrees of freedom of the system, e.g. $n = 3N$ if we are describing a system of $N \ge 1$ point-particles moving in 3-dimensional space. The motion of the system tends to minimize the potential energy, thus the system is at rest whenever its "state" (i.e. position) $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is located at a local minimum of $F$. These local minima are called **stable equilibria** of the system. There may also be **unstable equilibria**, namely critical points $x \in \mathbb{R}^n$ of $F$ that are not local minima: the equations of motion dictate that if $x$ lies *exactly* at such a point, then it will stay there, but the equilibrium is unstable in the sense that arbitrarily small perturbations of the state away from such a critical point can cause it to migrate further away, towards a different equilibrium point. The points of greatest interest are therefore the stable equilibria, i.e. the local minima of $F$. If you are given just a single potential function $F : \mathbb{R}^n \to \mathbb{R}$ and remember the methods you learned in *Analysis II* for identifying the local minima of such a function, then there is not much more to be said.

However, one often encounters mechanical systems that are described not just by a single potential but by a *family* of them, which depend smoothly on a finite set of adjustable parameters. Formally, this means considering a smooth function of the form

$$\mathbb{R}^n \times \mathbb{R}^p \xrightarrow{F} \mathbb{R} : (x, u) \mapsto F_u(x) := F(x; u),$$

in which we regard $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ as representing the state of the system and $u = (u_1, \dots, u_p) \in \mathbb{R}^p$ as the set of external parameters on which the system depends, so that each specific value of $u \in \mathbb{R}^p$ defines a different potential function $F_u : \mathbb{R}^n \to \mathbb{R}$ whose stable equilibria we would like to understand.[3] The following questions now arise naturally:

1. Under what conditions do the stable equilibria of $F_u$ vary continuously (and smoothly?) with $u$?
2. If they do not, then why not, and what happens instead?

[3]The use of a semicolon instead of a comma between $x$ and $u$ is meant to emphasize the distinction in meaning between the two sets of variables $x_1, \dots, x_n$ and $u_1, \dots, u_p$. This useful notational convention is not universal, but it is consistent with [Mon21].

## The cusp catastrophe

To see why these might be interesting questions, consider the following concrete example of a family of potentials $G_{u,v} : \mathbb{R} \to \mathbb{R}$ describing a system with one degree of freedom dependent on two external parameters $u, v \in \mathbb{R}$:

$$G(x; u, v) = G_{u,v}(x) := x^4 - ux^2 + vx.$$

This family of functions—or more precisely, the restriction of $G : \mathbb{R}^3 \to \mathbb{R}$ to arbitrarily small neighborhoods of the origin $(x, u, v) = 0 \in \mathbb{R}^3$—is known as the **cusp catastrophe**, and it is the second simplest in Thom's famous list of seven elementary catastrophes. It also serves as a local model for certain mechanical systems that are not hard to observe in nature of one makes an effort, e.g. an amusing gadget known as *Zeeman's catastrophe machine*, which can be built out of legos and rubber bands.[4]

You can use methods you learned in first-year analysis to see what happens to the set of critical points of $G_{u,v}$ as you move the parameters $(u, v) \in \mathbb{R}^2$ around in the plane, and it is an amusing exercise. The set of all critical points for all possible values of the parameters forms a surface

$$C_G := \left\{(x, u, v) \;\middle|\; G'_{u,v}(x) = 4x^3 - 2ux + v = 0\right\} \subset \mathbb{R}^3$$

called the **catastrophe set**, a picture of which is shown in Figure 1. A useful way to think about this picture is to consider various specific values of $(u, v) \in \mathbb{R}^2$ and ask how many critical points the function $G_{u,v} : \mathbb{R} \to \mathbb{R}$ has, or equivalently, how many points of the surface $C_G$ are in the preimage of a given point $(u, v) \in \mathbb{R}^2$ under the projection

$$C_G \to \mathbb{R}^2 : (x, u, v) \mapsto (u, v).$$

The simplest case is $u = v = 0$, for which the potential is just $G_{0,0}(x) = x^4$, which has only one critical point, a local minimum. This potential, however, is not **stable**, a notion that we will define more precisely when we develop the basic notions of catastrophe theory: in the present example, it means concretely that the set of equilibria can undergo sudden changes if you move the parameters $(u, v) \in \mathbb{R}^2$ slightly away from the origin. You will notice, for instance, that if you hold $v$ fixed at 0 but move $u$ slightly up to a positive value, then $G_{u,0}$ suddenly has not just one but *three* critical points: two local minima, and a local maximum between them. (You will easily see this if you observe that $G'_{u,v}$ is a cubic polynomial, and think about what graphs of cubic polynomials can look like in general.) Having perturbed the system in this way, it is now stable, meaning that if you move $(u, v)$ slightly from its new position, all three of the critical points will continue to exist, though they will also move slightly, in a way that depends smoothly on $(u, v)$. You can prove all this using methods of first-year analysis—mainly the implicit function theorem—the crucial detail being that the *second* derivative $G''_{u,v}(x)$ is nonzero at each of the critical points.

But you'll see another odd thing happen if you now hold $u > 0$ fixed and move $v$ a sufficiently large distance either up or down: the effect of this change is to move the graph of the cubic polynomial $G'_{u,v}(x) = 4x^3 - 2ux + v$ up or down, with the eventual outcome that $G_{u,v}$ again has only one critical point, necessarily a local minimum. Somewhere along the way, a transition occurs that allows three critical points to become only one, and such transitions are called **bifurcations**. What you'll find if you work out the details in this example is that as the bifurcation is approached, the local maximum and one of the local minima move toward each other until, at some special parameter value $(u, v)$ where the bifurcation happens, $G_{u,v}$ has only two critical points, one of them a so-called **degenerate** critical point, which is neither maximum nor minimum, and the second derivative at that point is 0. The degeneracy of this critical point permits a strange thing to happen when the parameter is moved further: the degenerate critical point simply disappears, leaving the other local minimum (which was not involved in the bifurcation) as the only remaining critical point. This sudden change sounds at first like a discontinuity, but in fact, only smooth functions are involved, and it is easy to see what might happen if you try sketching graphs of $G_{u,v}$: at the point of bifurcation, $G_{u,v}$ is a strictly increasing function near one of its critical points, and moving the parameter beyond the bifurcation causes its derivative near that point to increase, so that a nearby critical point is no longer possible. (See frames 6 and 10 of the "animation" in Figure 2.) But now imagine what must actually happen if the state

[4]For a recipe and a rough explanation of what Zeeman's catastrophe machine has to do with the specific function $G$, see [CH04, pp. 45-46].

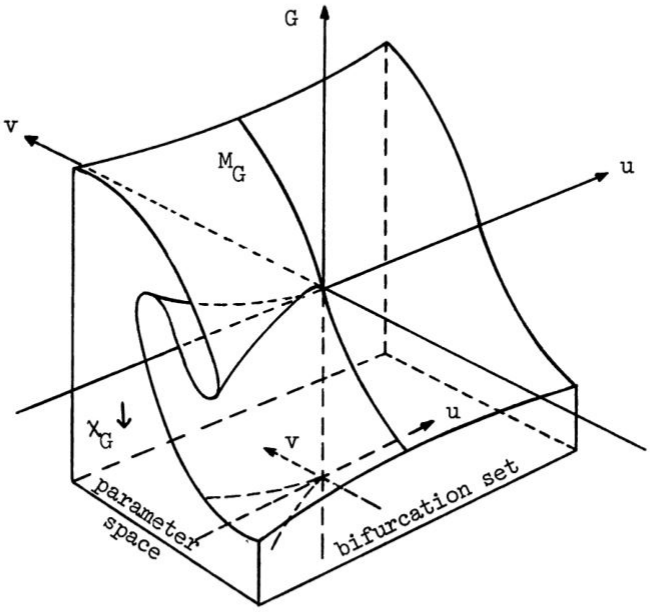


Figure 1: The catastrophe set $C_G \subset \mathbb{R}^3$ for the family of functions $G_{u,v} : \mathbb{R} \to \mathbb{R}$ defining the cusp catastrophe. (Picture copied from [Lu76, p. 107], where the catastrophe set $C_G$ is denoted by $M_G$.)

$x \in \mathbb{R}$ of the system lies at the stable equilibrium that disappears in the bifurcation. Figure 2 illustrates the phenomenon with a small ball that is shown following one of the stable equilibria as the system evolves. Before the bifurcation, the equilibrium moves smoothly with the parameter, but when the bifurcation is reached and this equilibrium disappears, the state has no choice but to seek out the other equilibrium point, thus forcing it to jump discontinuously. This sudden change is the "catastrophe" after which this branch of mathematics is named.

Now look again at Figure 1, and notice the dotted curves that separate the $uv$-plane into two regions, one of them a wedge-shaped region with a pointed end at the origin. The portion of the catastrophe set $C_G$ lying over the interior of this wedge-shaped region has three points $(x, u, v) \in C_G$ for each fixed value of the parameters $(u, v) \in \mathbb{R}^2$, while in the interior of the other region, there is only one $(x, u, v) \in C_G$ for each $(u, v) \in \mathbb{R}^2$. The border between the two regions of the parameter space is the so-called **discriminant**, or **bifurcation set**,

$$\Delta_G := \big\{(u, v) \bigm| G_{u,v} \text{ has a degenerate critical point}\big\} \subset \mathbb{R}^2.$$

In the picture, you can recognize the discriminant from the shape of the catastrophe set $C_G$, because over these points, the surface folds over itself, producing a total of two points $(x, u, v) \in C_G$ for each $(u, v) \in \mathbb{R}^2$ instead of one or three.

## Nondegeneracy and finite determinacy

The example of the cusp catastrophe contains simple special cases of several important concepts that will deserve our attention in this seminar. One is the notion of a *degenerate* critical point, which for a function $f : \mathbb{R} \to \mathbb{R}$ of one variable, just means a point $x \in \mathbb{R}$ at which both $f'(x)$ and $f''(x)$ vanish. For smooth functions $f : \mathbb{R}^n \to \mathbb{R}$, $x \in \mathbb{R}^n$ is a critical point if and only if $\nabla f(x) = 0$, and it is a **nondegenerate** critical

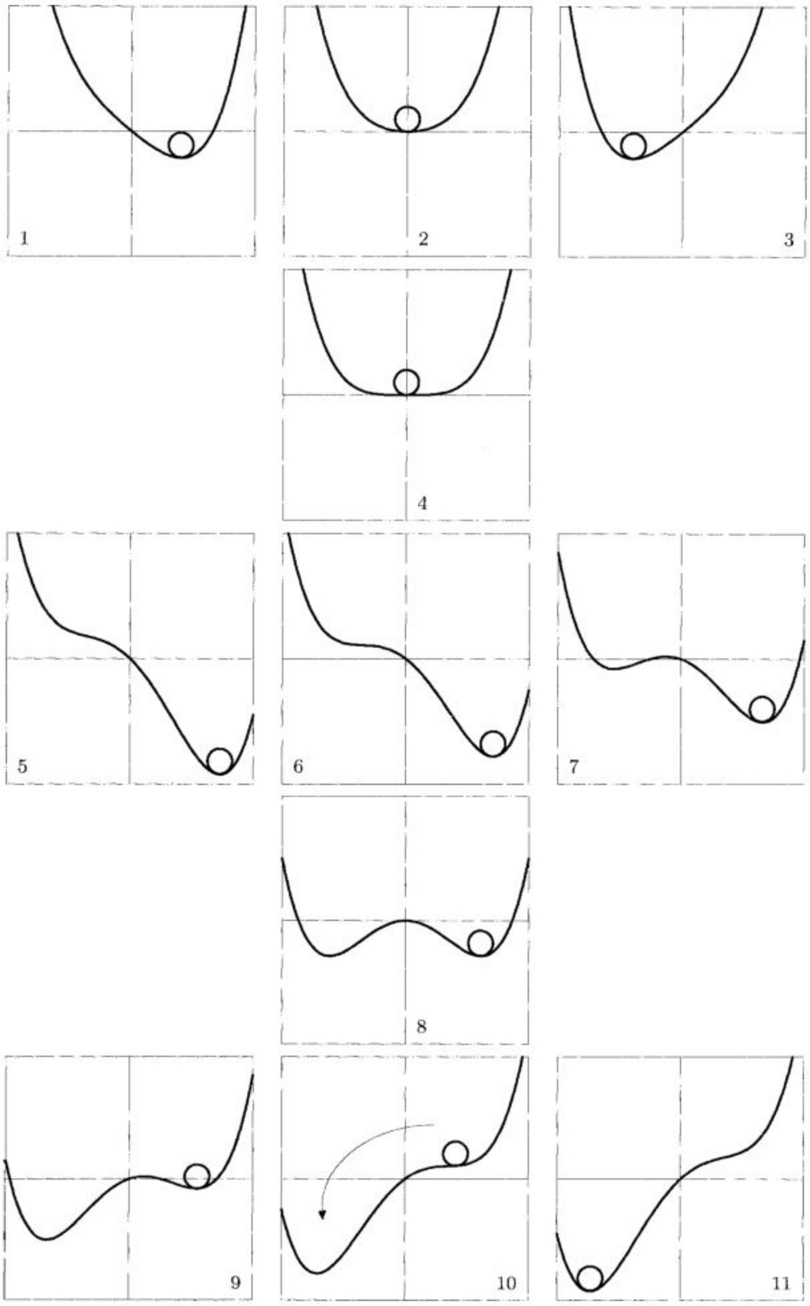


Figure 2: Evolution of the graph of the potential $G_{u,v}$ as the parameter $v \in \mathbb{R}$ moves while $u > 0$ is fixed. Bifurcations occur in frames 6 and 10, changing the total number of critical points. The small circle represents an object occupying one of the stable equilibrium states, which moves smoothly until that state becomes degenerate in frame 10 and then disappears, causing a catastrophe. (Pictures copied from [CH04, p. 44].)

point if additionally its *Hessian*

$$D(\nabla f)(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(x) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n}(x) \end{pmatrix} : \mathbb{R}^n \to \mathbb{R}^n$$

is an invertible linear map, or equivalently, a nondegenerate quadratic form. Another way to say this is that a critical point $x \in \mathbb{R}^n$ of $f : \mathbb{R}^n \to \mathbb{R}$ is nondegenerate if the gradient vector field $\nabla f$, regarded in its own right as a smooth map $\mathbb{R}^n \to \mathbb{R}^n$ that vanishes at $x$, does not have a *singularity* at $x$. This means in particular that $\nabla f$ satisfies the conditions of the inverse/implicit function theorem at $x$, and one can deduce from this two important properties of nondegenerate critical points:

1. They are *isolated*, meaning no other point in some neighborhood of a nondegenerate critical point is critical;
2. They are *stable*, meaning that for any small neighborhood of a nondegenerate critical point, any sufficiently small[5] perturbation of the function $f$ will also have a critical point in that neighborhood, in fact a unique one, which will also be nondegenerate.

With the goals of singularity theory in mind, the nondegenerate critical points of a function $f : \mathbb{R}^n \to \mathbb{R}$ have another property that must be mentioned: they each match one of exactly $n+1$ local models that can be written down precisely! This result is known as the **Morse lemma**, and what it says more precisely is that if $x \in \mathbb{R}^n$ is a nondegenerate critical point of $f : \mathbb{R}^n \to \mathbb{R}$ with $f(x) = 0$, then there exist smooth local coordinates $(x_1, \ldots, x_n)$ on some neighborhood of $x$, identifying $x$ with the origin in $\mathbb{R}^n$, such that $f$ is given in these coordinates by the formula

$$f(x_1, \ldots, x_n) = -x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_n^2$$

for some $k \in \{0, \ldots, n\}$. The number $k$ appearing in this formula is called the **index** of the critical point, and one can show that it does not depend on the choice of coordinates—its value can be deduced in fact entirely from the Hessian $D(\nabla f)(x)$. On the other hand, different critical points will generally have different indices, and the index is what determines the qualitative behavior of $f$ near such a point, e.g. the critical point is a local minimum if $k = 0$, a local maximum if $k = n$, or one of $n-1$ distinct types of saddle points for $k = 1, \ldots, n-1$. Since the index is determined by the Hessian, the Morse lemma tells us something quite nonobvious about this particular class of singularity: up to change of coordinates on small neighborhoods of a nondegenerate critical point $x$, the function $f$ is completely *determined by its Taylor expansion up to order* 2 at $x$. The Morse lemma thus achieves one of the holy grails of singularity theory: it allows us to determine (up to change of coordinates) the precise structure of a function near a singularity, based on knowledge of only *finitely many* of its derivatives at that one point. The general term for this is **finite determinacy**, and it is a property that not all singularities have, but we will see that the most important ones do.

## Unfoldings

One last remark: as you might extrapolate from the example of the cusp catastrophe, the most interesting aspects of singularity theory arise when one considers not just a single map $f : \mathbb{R}^n \to \mathbb{R}^m$ with its singularities,

[5] I am trying to keep this discussion non-technical, but strictly speaking, it would be important at this juncture to say more precisely what kind of perturbation of a function $f$ is considered "small," i.e. which metric or topology we are using on the space of functions $f : \mathbb{R}^n \to \mathbb{R}$. For present purposes, the $C^2_{\text{loc}}$-topology would suffice, meaning that we only need the values of the perturbed function and its derivatives up to second order to be close to those of $f$ on some neighborhood of the critical point. In this seminar, we will focus almost exclusively on *smooth* functions, which makes it more natural to use the $C^\infty_{\text{loc}}$-topology, whose definition you can infer by extrapolation from what was said above if you have not seen it before. Using the $C^\infty_{\text{loc}}$-topology produces a stronger condition than is necessary, but it is certainly sufficient, and there is no need to mention $C^k$ for any $k < \infty$ as long as we have no intention of studying finitely-differentiable functions.

but the ways that these singularities can change (continuously or otherwise) when $f$ is just one element in a smooth family of maps

$$\mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m : (x, u) \mapsto f_u(x), \qquad \text{such that} \qquad f_0 = f.$$

Such a family is called an **unfolding** of $f$, so for instance, the family $G_{u,v} : \mathbb{R} \to \mathbb{R}$ considered above was a (2-parameter) unfolding of the function $g(x) := G_{0,0}(x) = x^4$. Once we get far enough into the theory, we will be able to show that this particular unfolding has a special property: it is **versal**, which means essentially that on neighborhoods of the singularity of $x^4$, all other possible unfoldings of that function—with an arbitrary finite number of parameters—can be expressed in terms of that particular unfolding, which therefore serves as a model for all possible changes in the local structure of that singularity. We will see that one can use tools from commutative algebra to deduce which singularities admit versal unfoldings and how many parameters are required, and to identify whether a given unfolding is versal. This and finite determinacy together furnish the main ingredients needed for Thom's classification of the seven elementary catastrophes.

## Literature

In this seminar we will mainly follow the recent book by Montaldi [Mon21], which covers the essentials of catastrophe theory from a modern perspective in about the first 100 pages, and then builds upon this foundation to discuss singularity and bifurcation theory more generally. The book assumes an advanced undergraduate-level background in analysis and linear algebra, plus some rudimentary familiarity with rings and ideals; some appendices are also included for reviewing the essential background material from analysis and algebra. While [Mon21] develops most of the theory in full rigorous detail, it leaves one or two difficult details as black boxes, or in some cases, presents them in the wrong order, e.g. by postponing the proofs of certain results until after their main applications are worked out. Usually when Montaldi does this, there seem to be good pedagogical reasons for it, so we will mostly do the same in the seminar. The major exception is the proof of the Malgrange-Mather preparation theorem, for which we will need to consult some of the other sources mentioned below, because the result is too important to leave unproved.

The books by Castrigiano-Hayes [CH04], Bröcker [Brö75] and Lu [Lu76] are all somewhat older but also give well-written presentations of the standard material from catastrophe theory and related subjects. For certain slightly more specialized material, you may also have occasion to look at Arnold et al [AGZV12] and (for bifurcation theory especially) Golubitsky et al [GS85, GSS88].

All of these books are available in electronic form through the HU library from within the HU network. The following caveat applies specifically to Montaldi: the electronic access granted to us by Cambridge University Press is somewhat inconvenient to use, e.g. you cannot download a PDF, but can only read it online through a web interface or on your own computer (without internet connection) using special e-reader software that you can download from CUP and install. I've found that reading it online is a bad idea, because the figures and tables refuse to load half the time; downloading the e-reader works better, but aside from the annoyance in principle of being forced to install special software, the e-reader does not work as efficiently as a simple PDF reader, and it provides numerous extra features that you are unlikely to care about. If it were not a good book, I might refuse to use it just out of annoyance about this situation, but in fact, the book is pretty good, and it is not even unthinkable that you might decide to shell out 50€ for the printed paperback edition.

In theory, there should also exist one or two decent books in German, and I have a perhaps hallucinatory memory that I have once seen the original German version of [Brö75], but I cannot find it now and have no idea where to look. One thing this means is that if you choose to give a talk in German (which is allowed), you will likely be on your own for coming up with reasonable German equivalents of the English terminology. (I would love to know what Bröcker called an "unfolding" when he first taught the course on which his book was based—plausible conjuctures are welcome.)

## Plan of talks

Topics marked with an asterisk (like this*) in the following are more challenging than the others, and it would make sense for those talks to be given by Master- instead of Bachelor-students. For most of the talks, I have given a list of specific topics that should be considered essential to cover. I have not included *examples* on those lists, but as a general rule, it would be a good idea to include as many examples as you have time for in addition to the essential topics. This is also a subject in which well-drawn pictures can communicate quite a lot, so you should plan to include some wherever appropriate, but you may need to practice them a nontrivial amount in advance.

1. April 19: *Introduction and planning of further talks*
   I will give a general introduction to the subject, including much of what is discussed above and the relevant definitions from Chapter 2 of Montaldi. We will then distribute future talks among the participants. (The exact details of how we do this will depend on how many participants there are, but I plan on fixing speakers for at least the next two talks at this meeting.)

2. April 26: *The ring of germs of smooth functions* (Montaldi, Chapter 3)
   Essential topics to cover:

   - definitions of the ring $\mathcal{E}_n$ of smooth function-germs $(\mathbb{R}^n, 0) \to \mathbb{R}$, the ideals $\mathfrak{m}_n$, $\mathfrak{m}_n^2$, $\mathfrak{m}_n^3$,... and their analytical meaning
   - Hadamard's lemma
   - Newton diagrams
   - Nakayama's lemma and its application to studying ideals of finite codimension

   The contents of §3.6 should be considered optional and can be skipped if there is insufficient time. The most important thing in this chapter is Nakayama's lemma, which will be used over and over again in subsequent developments. Montaldi outsources its proof to Appendix D.7, which presents it in the more general context of modules and submodules; it may not be necessary to present this proof in full detail, but you should at least try to communicate the general idea behind it.
   *Suggestion*: While the participant giving this talk will only have a week to prepare it, it should be quite easy for someone who enjoys algebra.

3. May 3: *Right equivalence and the splitting lemma* (Montaldi, Chapter 4)
   Essential topics to cover:

   - definitions of $\mathcal{R}$-equivalence and $\mathcal{R}^+$-equivalence for two function-germs $(\mathbb{R}^n, 0) \to \mathbb{R}$
   - definition of the Jacobian ideal of a function-germ
   - the codimension of a germ $f \in \mathfrak{m}_n^2$
   - nondegenerate critical points, characterization in terms of the Jacobian ideal (Proposition 4.9)
   - Morse lemma and outline of proof
   - Splitting lemma (Theorem 4.13 and Corollary 4.15)

   Two things in this talk will be absolutely indispensable for understanding the rest of the seminar: the notion of the *codimension* of a singularity, and the splitting lemma. Montaldi characterizes his proof of both this and the Morse lemma as an "outline," which means that you should try to understand and convey the main ideas but not stress too much if not every detail seems completely clear. Any gaps that arise are meant to be filled in by the results about finite determinacy in Chapter 5.

4. May 10: *Finite determinacy** (Montaldi, Chapter 5)
   Essential topics to cover:

   - definition of the module $\theta_n$ of germs of smooth vector fields
   - definition of the tangent map $tf$ of a smooth map $f : \mathbb{R}^n \rightarrowtail \mathbb{R}^p$ or map-germ $f : (\mathbb{R}^n, 0) \to \mathbb{R}^p$, relation to the Jacobian ideal

- the Thom-Levine principle (Theorem 5.4)
- definition of the right tangent space $T\mathcal{R} \cdot f$ of a function-germ $f \in \mathcal{E}_n$
- statement and easy corollaries of the finite determinacy theorem (Theorem 5.10)
- partial converse of the finite determinacy theorem (statement without proof)
- proof of finite determinacy via the homotopy method

Montaldi's proof of the finite determinacy theorem includes a preparatory section explaining how the same "homotopy method" can be used to give a quite elegant proof of the classical inverse function theorem; it then uses a similar argument to prove finite determinacy but skips some details. For the talk, the proof of the inverse function theorem could be skipped, but you will have to make a judgement as to whether doing so makes your total presentation shorter or longer (it is not obvious to me). The contents of §5.7 should be considered optional.
*Suggestion*: While Montaldi's presentation in this chapter does not make any direct use of the theory of smooth manifolds, you may notice if you already know some differential geometry that that is where a lot of the ideas behind the proof come from, and this will make it easier to understand the big picture. I would therefore suggest that someone who likes manifolds should give this talk.

5. May 17: *The elementary catastrophes* (Montaldi, Chapter 6)
This chapter is not very long, and it should be possible to cover all of it (I would consider the contents of §6.4 optional, in any case). The point is to apply the machinery developed so far toward classifying (up to right equivalence) the simplest singularities for function-germs $f : (\mathbb{R}^n, 0) \to \mathbb{R}$. One technical detail: if Proposition 4.16 (which appears at the very end of Chapter 4) was not already proved in the May 3 talk, then you'll have to cover it here.

6. May 24: *Unfoldings* (Montaldi, Chapter 7)
Essential topics to cover:

- $C_F \to \mathbb{R}^p : (x, u) \mapsto u$ is a local diffeomorphism away from the discriminant (Proposition 7.1)
- induced unfoldings (Definition 7.3)
- the equivalence relation for unfoldings
- versality and its algebraic characterization (Theorem 7.8; statement without proof)[6]
- examples of versal unfoldings (as many as there is time for)
- the list of elementary catastrophes

The person who gives this talk is getting away with murder, because they will get to introduce a lot of fun concepts and draw cool pictures, but the proof of the most important theorem in this chapter (on the algebraic characterization of versal unfoldings) will be postponed until the next talk. The relatively short discussion of *simple* singularities in §7.6 would be nice to include, but can be considered optional; §7.7 is even more optional.
*Caution*: Some rudimentary understanding of smooth maps between manifolds is required in this talk, e.g. in order to understand what it means to say that $C_F \to \mathbb{R}^p : (x, u) \mapsto u$ is a local diffeomorphism.

7. May 31: *The Malgrange-Mather preparation theorem** (§16.1 and §16.3 in Montaldi, then [CH04, Chapter 9] or [Brö75, Chapter 6])
Essential topics to cover:

- Statement of the theorem (Montaldi §16.1)
- Application to the proof of Theorem 7.8 (Montaldi §16.3)
- As much of the proof of the theorem as you can fit, following e.g. [CH04] or [Brö75]

[6] A note on terminology: most of the older sources such as [Brö75, CH04, GS85, GSS88] call an unfolding **universal** if it has the minimal number of parameters required for versality. Somewhere along the line, somebody must have complained that this use of the word "universal" is slightly (though in my opinion not horribly) inconsistent with its standard meaning in category theory, so the word **miniversal** was created to replace it, and is used in [Mon21, AGZV12]. I have not yet decided which side I am on.

The major weakness of Montaldi's book is that it does not include the proof of this technical result, on which the proofs of arguably the most important theorems in singularity theory are based. Even the statement is postponed until relatively late in the book, because it is used not only for versal unfoldings in catastrophe theory, but also for much more general results of a similar nature in wider singularity-theoretic contexts. In any case, Montaldi's discussion of the statement and how to apply it are worthwhile; I would recommend skipping §16.2 and §16.4 (unless you find some of the examples in §16.2 especially helpful for understanding the statement), and focusing at first on the proof of Theorem 7.8 that appears in §16.3. One should then budget at least the second half of the talk for proving (or at least sketching the proof of) the preparation theorem, for which you will need to consult a different book, such as Castrigiano-Hayes or Bröcker.

8. June 7: *Plane curve and hypersurface singularities* (Montaldi, Chapter 8)
   This chapter explores further applications of the theory developed so far, and will give the speaker an excuse to draw a lot of pictures. It should be possible to cover all of it.

9. June 14: *Catastrophes with symmetry* (Montaldi, Chapter 9 and §16.2 and §16.4)
   Chapter 9 appears shorter than a full talk, until you notice that it outsources an important technical result to Chapter 16, where it uses the Malgrange preparation theorem; it should be possible in any case to cover all of Chapter 9 plus the two sections of Chapter 16 that are related to it. Chapter 9 is intended as a hint toward a much larger and very important topic, namely how to modify the general machinery of singularity theory for situations in which one is only interested in maps that obey given symmetries, and how the resulting classifications of singularities can change in that setting. If you find this topic interesting and want to read some more about where it leads, I recommend taking a look at [GSS88].

10. June 21: *Bifurcation problems and contact equivalence* (Montaldi, Chapters 10 and 11)
    Essential topics to cover:

    - definitions of the terms *bifurcation problem*, *singular set* and (in that context) *discriminant* (from Chapter 10)
    - definitions of $\mathcal{K}$-equivalence and $\mathcal{C}$-equivalence for map-germs $(\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$
    - definition of the *local algebra* $Q(f) = \mathcal{E}_n / I_f$ of a map-germ $f$
    - algebraic characterizations of $\mathcal{C}$- and $\mathcal{K}$-equivalence (Theorem 11.5 and Corollary 11.7)
    - algebraic and geometric multiplicities of a map-germ
    - the alternative definition of $\mathcal{K}$-equivalence, and why it is not different

    At this point in the seminar, we are finished with catastrophe theory per se and beginning to explore singularity and bifurcation theory more generally, with a focus on understanding the local structure of solution sets $f^{-1}(0)$ defined by smooth map-germs $(\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$. The notion of right equivalence is too simplistic for this more general setting, so the first step is to establish a more appropriate equivalence relation for map-germs, and that is the main subject of this talk.
    *Suggestion*: This is another talk in which knowledge of differential geometry is not strictly required, but would be helpful for the sake of intuition. In particular, my own favorite way of understanding contact equivalence is in terms of vector bundles: one can define two germs of smooth sections of vector bundles to be $\mathcal{K}$-equivalent if there exist choices of local coordinates and local trivializations in which they look identical—specializing this to a situation where the vector bundles are given with local trivializations (and sections are therefore locally equivalent to smooth maps $\mathbb{R}^n \to \mathbb{R}^p$) essentially reproduces the definition given by Montaldi. If you generalize this picture from vector bundles to smooth fiber bundles (whose fibers are manifolds but not necessarily vector spaces), you obtain the seemingly more general notion of $\mathcal{K}'$-equivalence, and a linearization argument explains why it is actually the same thing as $\mathcal{K}$-equivalence.

11. June 28: *Tangent spaces of equivalence classes** (Montaldi, Chapter 12, excluding §12.9)
    Essential topics to cover:

- the $\mathcal{E}_n$-modules $\mathcal{E}_n^p$ of smooth map-germs $f : (\mathbb{R}^n, 0) \to \mathbb{R}^p$, vector fields $\theta_n$, and vector fields $\theta_n(f)$ along $f \in \mathcal{E}_n^p$, plus the Jacobian module $Jf \subset \theta(f)$ of $f \in \mathcal{E}_n^p$, and submodules of interest
- the right tangent space $T\mathcal{R} \cdot f$ and extended right tangent space $T_e\mathcal{R} \cdot f$ of a map-germ $f \in \mathcal{E}_n^p$
- left equivalence and the left (and extended left) tangent spaces $T\mathcal{L} \cdot f$ and $T_e\mathcal{L} \cdot f$ of $f \in \mathcal{E}_n^p$
- left-right equivalence and the left-right tangent space $T\mathcal{A} \cdot f$ of $f \in \mathcal{E}_n^p$ (just to have seen the definitions, otherwise not much needs to be said about this for our purposes)
- the tangent and extended tangent space $T\mathcal{K} \cdot f$ of $f \in \mathcal{E}_n^p$ for contact equivalence, and the notion of $\mathcal{K}$-codimension
- statement (without proof) of the general theorem on finite codimension and finite determinacy (Theorem 12.4)
- contact equivalence for unfoldings and the tangent space $T_e\mathcal{K}_{\mathrm{un}} \cdot F$
- the general Thom-Levine principle (Theorem 12.8)

The Thom-Levine principle should be considered the central result and main objective in this talk. Otherwise, the chapter contains a large number of somewhat cumbersome definitions, but several of them are quite similar, and they all follow a particular philosophy that is not hard to explain; if stating all of them takes too long, then it is perhaps best to focus on those which specifically involve contact equivalence. This is yet another topic for which some knowledge of the theory of smooth manifolds is an advantage for understanding, though it is not used so explicitly.

12. July 5: *Finite determinacy for contact equivalence*$^*$ (Montaldi, §12.9 and §13.1)
The straightforward goal here is to apply ideas from the previous two talks toward proving Theorem 13.2, which gives a sufficient algebraic condition for a smooth map-germ to be $k$-determined with respect to contact equivalence, an essential ingredient for proving classification results.

13. July 12: *Classifying equidimensional map-germs of low codimension*$^*$ (Montaldi, §13.2)
I would leave this detail up to the speaker's discretion, but instead of presenting *all* of the classification results mentioned by Montaldi in this section, it would also be interesting to go into more detail about the Boardman symbol, which Montaldi mentions only briefly. It is discussed more extensively in [AGZV12, Chapter 2] and [Boa67].

14. July 19: *Sketch of bifurcation theory*$^*$ (Montaldi, Chapters 18–?)
At this point in the seminar, there is no time left to prove anything nontrivial, but one could give an interesting survey of some notions from bifurcation theory as discussed by Montaldi in Chapter 18, and then try to sketch a few of the technical details as covered in the subsequent chapters. Montaldi adopts the so-called "path approach" to bifurcation theory, in which a 1-parameter deformation of a given singularity is considered equivalent to a path through the parameter space of its versal unfolding. This is slightly different from the older "distinguished parameter approach" found e.g. in [GS85, GSS88], and Montaldi spends a lot of effort showing that the two approaches lead to the same classification results for 1-parameter bifurcations, a conclusion that may seem underwhelming if you haven't read [GS85, GSS88] (which are worth reading, by the way). The path approach seems to be more versatile, but also presents some technical challenges that the distinguished parameter approach does not, arising e.g. from the fact that discriminants are typically algebraic subsets but not smooth submanifolds—dealing with these technical challenges is the purpose of Montaldi's Chapters 19 and 20. For the purposes of a quick survey, a reasonable approach might be to adopt the convenient fiction that discriminants are always smooth submanifolds, which ought to make Chapter 19 superfluous.

## Otherwise

All other practical information such as the location and an up-to-date schedule for the seminar is posted on the seminar webpage at

http://www.mathematik.hu-berlin.de/~wendl/Sommer2024/Katastrophen/

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