Differential Geometry I — Humboldt-Universität zu Berlin
C. Wendl — Winter Semester 2016–17

PROBLEM SET 10

Organizational note

For the next two weeks after this, the usual weekly problem sets will be replaced by a take-home midterm, which will be distributed January 10 and due January 24. There will be no lecture or problem class on January 17.

Suggested reading

Lecture notes (on the website): Chapter 3, *Connections*

Problems

1. Given a smooth n-manifold M and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, denote by $\Omega^1(M, \mathbb{F}^{m\times m})$ the space of matrix-valued 1-forms, i.e. smooth maps $TM \to \mathbb{F}^{m\times m}$ that are real-linear on each fiber of TM. If $\pi : E \to M$ is a smooth vector bundle with a local trivialization $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{F}^m$, we shall write $\Phi_\alpha(v) = (x, v_\alpha)$ for $v \in E_x$ and $x \in \mathcal{U}_\alpha$, hence smooth sections $s \in \Gamma(E|_{\mathcal{U}_\alpha})$ correspond to smooth vector-valued functions $s_\alpha : \mathcal{U}_\alpha \to \mathbb{F}^m$. Any choice of matrix-valued 1-form $A_\alpha \in \Omega^1(\mathcal{U}_\alpha, \mathbb{F}^{m\times m})$ can then be used to define a linear map
$$\Gamma(E|_{\mathcal{U}_\alpha}) \to \Gamma(\mathrm{Hom}(TM, E)|_{\mathcal{U}_\alpha}) : s \mapsto \nabla s$$
such that for any $x \in \mathcal{U}_\alpha$ and $X \in T_xM$, $\nabla_X s := \nabla s(X) \in E_x$ satisfies
$$(\nabla_X s)_\alpha = ds_\alpha(X) + A_\alpha(X)s_\alpha(x). \tag{1}$$
Now suppose $\Phi_\beta : E|_{\mathcal{U}_\beta} \to \mathcal{U}_\beta \times \mathbb{F}^m$ is a second local trivialization related to Φ_α by $\Phi_\beta \circ \Phi_\alpha^{-1}(x, v) = (x, g_{\beta\alpha}(x)v)$ for a smooth transition function $g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{GL}(m, \mathbb{F})$. Show that there exists $A_\beta \in \Omega^1(\mathcal{U}_\alpha \cap \mathcal{U}_\beta, \mathbb{F}^{m\times m})$ such that $(\nabla_X s)_\beta = ds_\beta(X) + A_\beta(X)s_\beta(x)$ also holds when $x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$, and derive the so-called *gauge transformation* formula relating A_α and A_β,
$$A_\alpha = g_{\beta\alpha}^{-1} A_\beta g_{\beta\alpha} + g_{\beta\alpha}^{-1}\, dg_{\beta\alpha}. \tag{2}$$
Note that each of the three terms in this expression should be understood as a matrix-valued 1-form, e.g. for $X \in T_xM$, the first term on the right hand side maps X to the matrix $[g_{\beta\alpha}(x)]^{-1} A_\beta(X) g_{\beta\alpha}(x)$, which is a product of three matrices.

2. Problem 1 shows that for a section s of a vector bundle $\pi : E \to M$, one cannot generally define directional derivatives $\nabla_X s$ with respect to tangent vectors $X \in T_xM$ simply by choosing a local trivialization Φ_α near x and computing the derivative in coordinates: this would amount to setting $A_\alpha \equiv 0$ in (1), but the resulting notion depends on the choice since by (2), A_β will not usually vanish unless the transition function happens to be constant. Observe however that if x is a point where $s(x) = 0$, then we can ignore the matrix-valued 1-forms at x and just write $(\nabla_X s)_\alpha = ds_\alpha(X)$ and $(\nabla_X s)_\beta = ds_\beta(X)$. This hints at the fact that whenever $s(x) = 0$, there is a well-defined *linearization*
$$Ds(x) : T_xM \to E_x$$
which can be defined as the linear map $X \mapsto \nabla_X s$ given by *any* choice of connection ∇, and it does not depend on this choice.

   (a) Prove the above claim that $Ds(x) := \nabla_X s$ does not depend on the choice of connection if $s(x) = 0$. Do it without using local trivializations or coordinates.
   Hint: If $p \in E$ is a point in the zero-section, what might the horizontal subspace H_pE defined by a connection look like?

(b) We say that a section $s \in \Gamma(E)$ is *transverse to the zero-section* if the linearization $Ds(x) : T_x M \to E_x$ is surjective for every $x \in s^{-1}(0)$. Show that under this condition, the zero-set $s^{-1}(0)$ is a smooth submanifold of M. What is its codimension?

3. For this problem, take the word "connection" on a vector bundle $\pi : E \to M$ to mean a linear map $\nabla : \Gamma(E) \to \Gamma(\mathrm{Hom}(TM, E))$ which satisfies the Leibniz rule

$$\nabla(fs)(X) = df(X)s + f\,\nabla s(X)$$

for all $f \in C^\infty(M)$, $s \in \Gamma(E)$ and $X \in TM$. Use C^∞-linearity to show that if ∇ and $\widetilde{\nabla}$ are any two connections in this sense, then there exists a smooth bundle map $A : TM \oplus E \to E$ such that for all $s \in \Gamma(E)$ and $X \in TM$,

$$\nabla s(X) = \widetilde{\nabla} s(X) + A(X, s).$$

4. Define S^{2n-1} as the unit sphere in $\mathbb{C}^n$ and observe that unitary linear transformations $\mathbf{A} \in \mathrm{U}(n)$ map S^{2n-1} to itself. Then if $\mathbf{e}_1 = (1, 0, \ldots, 0) \in \mathbb{C}^n$, define the map

$$\pi : \mathrm{U}(n) \to S^{2n-1} : \mathbf{A} \mapsto \mathbf{A}\mathbf{e}_1.$$

(a) Given a matrix $\mathbf{B} \in \mathrm{U}(n-1)$ identify this with the slightly larger matrix

$$\begin{pmatrix} 1 & \\ & \mathbf{B} \end{pmatrix} \in \mathrm{U}(n),$$

and show that the map

$$\mathrm{U}(n) \times \mathrm{U}(n-1) \to \mathrm{U}(n) : (\mathbf{A}, \mathbf{B}) \mapsto \mathbf{A}\mathbf{B}$$

defines a right action of $\mathrm{U}(n-1)$ on $\mathrm{U}(n)$ which preserves the level sets $\pi^{-1}(\mathbf{v})$ for $\mathbf{v} \in S^{2n-1}$.

(b) Show that π is surjective and for each $\mathbf{v} \in S^{2n-1}$, $\pi^{-1}(\mathbf{v})$ is a smooth manifold diffeomorphic to $\mathrm{U}(n-1)$.

(c) From the above considerations, it's not hard to believe that $\pi : \mathrm{U}(n) \to S^{2n-1}$ is a principal $\mathrm{U}(n-1)$-bundle: to prove this one must construct appropriate local trivializations. In light of the group action, it suffices in fact to construct local sections near each point, which isn't hard. Let's bypass this detail and ask instead the following question: given that $\pi : \mathrm{U}(n) \to S^{2n-1}$ is a principal $\mathrm{U}(n-1)$-bundle, is it the *frame bundle* of some Hermitian vector bundle of rank $n-1$? The answer is yes—identify the vector bundle in question, and explain.