Differential Geometry I
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PROBLEM SET 3

Suggested reading

As usual, chapter and section indications in Lee refer to the 2003 edition and may differ in the 2013 edition. (And unlike last week, I really mean it this time.)

- Baum: §2.5–2.6
- Lee: Chapter 7, Chapter 8 ("Embedded Submanifolds"), Chapter 11 (up to "Tensors and Tensor Fields on Manifolds")

For an overview of the multilinear algebra underlying tensors, you might also find Appendix A of my *Lecture Notes on Bundles and Connections* useful; it is now posted on the course website, along with a convenient link to Helga Baum's Volesungsskript.

Problems

1. The purpose of this problem is to unpack the meaning of a few definitions that we saw in last Thursday's lecture. Recall that a smooth map $f : M \to N$ is called an *immersion* or a *submersion* if its derivative $f_* : T_pM \to T_{f(p)}N$ at every point $p \in M$ is *injective* or *surjective* respectively. We say $f : M \to N$ is an *embeddeing* if it is an injective immersion that is a homeomorphism onto its image. The latter condition means in particular that $f^{-1} : f(M) \to M$ is continuous, where $f(M) \subset N$ is assumed to carry the topology (or metric if you prefer) that it inherits as a subset of N. We often use the notation

$$f : M \looparrowright N \quad \text{or} \quad f : M \hookrightarrow N$$

to indicate that f is an immersion or embedding respectively. Finally, a subset $M \subset N$ in a smooth manifold N is called a *submanifold of* N if it is a topological manifold (with the topology it inherits from N) and admits a smooth structure such that the natural inclusion $M \hookrightarrow N$ is an embedding.
Achtung! You will occasionally see subtle discrepancies between different books in some of these definitions. The conventions adopted here are consistent with both Lee and Baum, but slightly different from e.g. Warner. Lee also uses the term "immersed submanifold", which I prefer to avoid, but it is useful occasionally.

The following problem is likely to seem somewhat difficult, though most of the answers may be found in this week's suggested reading, so you can appeal to the books for help if necessary. But try it on your own first.

   (a) Suppose $\mathcal{U} \subset \mathbb{R}^m$ is an open subset and $f : \mathcal{U} \to \mathbb{R}^n$ is a smooth map with $df(x) : \mathbb{R}^m \to \mathbb{R}^n$ injective for some $x \in \mathcal{U}$. Find a smooth map $\tilde{f} : \mathcal{U} \times \mathbb{R}^{n-m} \to \mathbb{R}^n$ such that $d\tilde{f}(x) : \mathbb{R}^m \oplus \mathbb{R}^{n-m} \to \mathbb{R}^n$ is invertible.

   (b) Use problem (1a) and the inverse function theorem to prove that if $f : M \looparrowright N$ is an immersion between smooth manifolds, then every point $p \in M$ has an open neighborhood $\mathcal{U} \subset M$ such that $f|_{\mathcal{U}} : \mathcal{U} \hookrightarrow N$ is an embedding. We say in this case that f is a "local embedding".
   Hint: Since the statement is local and all manifolds can be described locally via charts, it suffices to consider the special case where $N = \mathbb{R}^n$ and M is an open subset of $\mathbb{R}^m$.

   (c) Find an example of an injective immersion $f : \mathbb{R} \looparrowright \mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ that is not an embedding.
   Hint: Define it so that its image is dense in $\mathbb{T}^2$, then convince yourself that $f^{-1} : f(\mathbb{R}) \to \mathbb{R}$ cannot be continuous with respect to the topology that $f(\mathbb{R})$ inherits from $\mathbb{T}^2$.

   (d) Find an example of an injective immersion $f : (-1, 1) \looparrowright \mathbb{R}^2$ that is not an embedding. *Hint:* ∞.

   (e) Show that a subset $M \subset N$ of a smooth manifold N is a submanifold if and only if it is the image of an embedding.

(f) Show that if $f : M \looparrowright N$ is an injective immersion and M is compact, then f is an embedding.

(g) If you are sufficiently ambitious, use what you learned in problems (1a) and (1b) to establish the following alternative characterization of submanifolds. Given a smooth n-manifold N, show that a subset $M \subset N$ is an m-dimensional submanifold of N if and only if for every $p \in M$ there exists an open neighborhood $\mathcal{U} \subset N$ of p and a chart $x : \mathcal{U} \to \mathbb{R}^n$ such that

$$M \cap \mathcal{U} = \left\{ q \in \mathcal{U} \;\middle|\; x(q) \in \mathbb{R}^m \times \{0\} \subset \mathbb{R}^n \right\}.$$

In other words, the local coordinates $x = (x^1, \ldots, x^n)$ characterize M near p as the set of all points whose last $n - m$ coordinates vanish. A chart with this property is called a *slice* chart. *Question for thought: In constructing slice charts for a given embedded submanifold $M \subset N$, where do you need to use the assumption that the inclusion $M \hookrightarrow N$ is a homoemorphism onto its image?*

Since the remainder of this problem set deals with tensors, we need to clarify some notational conventions that differ in various books. The notation that I use in class and in my lecture notes is as follows. If V is a vector space, V^k_ℓ denotes the vector space of tensors on V that are *covariant* of rank ℓ and *contravariant* of rank k, i.e. multilinear maps

$$\underbrace{V \times \ldots \times V}_{\ell} \times \underbrace{V^* \times \ldots \times V^*}_{k} \to \mathbb{R}$$

where we denote the dual space $V^* = \mathrm{Hom}(V, \mathbb{R}) = V^0_1$. Replacing V with a tangent space T_pM, we call the corresponding tensor space $(T^k_\ell M)_p$ and define the *tensor bundle* $T^k_\ell M$ to be the union of these spaces for all $p \in M$. In particular, $T^1_0 M$ is the usual tangent bundle TM, and $T^0_1 M$ is the *cotangent bundle*

$$T^*M := T^0_1 M,$$

made up of the dual spaces $T^*_p M$ to the tangent spaces T_pM. A *tensor of type* (k, ℓ) on T_pM is then simply an element of $(T^k_\ell M)_p$, and a *tensor field*[1] of type (k, ℓ) smoothly assigns to each $p \in M$ an element of $(T^k_\ell M)_p$. The space of tensor fields of this type is denoted $\Gamma(T^k_\ell M)$, literally, "sections of the bundle $T^k_\ell M$". We'll define precisely what *section* means when we discuss bundles in earnest later in the semester.

Here's a brief glossary of some notational differences between our discussion and that of Lee's book. Each choice has its own logic, and neither is perfect. You will find further differences in the book by Agricola-Friedrich and in Baum's Vorlesungsskript, e.g. what we call V^k_ℓ is called $T^{(\ell,k)}(V)$ by Baum.

our notation		**Lee's notation**
$V^0_k, T^0_k M, (T^0_k M)_p$	$=$	$T^k(V), T^k M, T^k(T_pM)$
$V^k_0, T^k_0 M, (T^k_0 M)_p$	$=$	$T_k(V), T_k M, T_k(T_pM)$
$V^k_\ell, T^k_\ell M, (T^k_\ell M)_p$	$=$	$T^\ell_k(V), T^\ell_k M, T^\ell_k(T_pM)$
tensor of type (k, ℓ)	$=$	tensor of type $\binom{\ell}{k}$

Yes, that's right, the k and ℓ are flipped—I swear it's not my fault. My logic (and I'm not the only one) is that the *covariant* tensors, i.e. those which act on vectors but not on dual vectors, should be indicated by a *lower* index because their components have lower indices. Similarly it makes sense to say $TM = T^1_0 M$ instead of $T^0_1 M$ because the components of tangent vectors have upper indices.

2. Let V and W be vector spaces, and for $k \in \mathbb{N}$ denote by $\mathrm{Hom}_k(V, W)$ the vector space of k-multilinear maps $\underbrace{V \times \ldots \times V}_{k} \to W$. Find a natural isomorphism $\mathrm{Hom}_k(V, V) \to V^1_k$, and prove that it is an isomorphism.
Note: there are multiple isomorphisms that one could write down, but only one that is truly natural. *To prove it's an isomorphism, remember it suffices to show that the map is linear and injective, and that* $\dim \mathrm{Hom}_k(V, V) = \dim V^1_k$. *What are these dimensions actually? If it simplifies things, you may as well assume $V = \mathbb{R}^n$.*

[1] One often omits the word "field" from "tensor field" when there's no danger of confusion.

3. Let M be an n-dimensional manifold with an open set $\mathcal{U} \subset M$ and coordinate chart $x = (x^1, \dots, x^n) : \mathcal{U} \to \mathbb{R}^n$. Recall that the coordinate functions $x^j : \mathcal{U} \to \mathbb{R}$ define derivations $\partial_j := \frac{\partial}{\partial x^j}$ and differentials dx^j, which give bases of T_pM and T_p^*M respectively at every point $p \in \mathcal{U}$. With these, an arbitrary tensor field $T \in \Gamma(T^k_\ell M)$ can be expressed over $\mathcal{U}$ via its $n^{k+\ell}$ *component functions* $T^{i_1\dots i_k}{}_{j_1\dots j_\ell} : \mathcal{U} \to \mathbb{R}$, defined by

$$T^{i_1\dots i_k}{}_{j_1\dots j_\ell} = T(\partial_{j_1}, \dots, \partial_{j_\ell}, dx^{i_1}, \dots, dx^{i_k}).$$

We then have

$$T = T^{i_1\dots i_k}{}_{j_1\dots j_\ell}\ dx^{j_1} \otimes \dots \otimes dx^{j_\ell} \otimes \partial_{i_1} \otimes \dots \otimes \partial_{i_k},$$

using the Einstein summation convention: recall that since this expression contains $k + \ell$ pairs of matching upper and lower indices, there's an implied summation over each one. Literally then (we'll write it out just this once), this means

$$T = \sum_{i_1=1}^{n} \dots \sum_{i_k=1}^{n} \sum_{j_1=1}^{n} \dots \sum_{j_\ell=1}^{n} T^{i_1\dots i_k}{}_{j_1\dots j_\ell}\ dx^{j_1} \otimes \dots \otimes dx^{j_\ell} \otimes \partial_{i_1} \otimes \dots \otimes \partial_{i_k}.$$

That's why we usually don't write it out literally.

(a) Consider a tensor field S of type $(3, 2)$ and another T of type $(2, 1)$, and recall that the *tensor product* $S \otimes T$ is then a tensor field of type $(5, 3)$ defined by

$$(S \otimes T)(X, Y, Z, \alpha, \beta, \gamma, \theta, \omega) = S(X, Y, \alpha, \beta, \gamma) \cdot T(Z, \theta, \omega)$$

for any tangent vectors $X, Y, Z \in T_pM$ and cotangent vectors $\alpha, \beta, \gamma, \theta, \omega \in T_p^*M$. Find a formula for the component functions $(S \otimes T)^{ijk\ell m}{}_{pqr} : \mathcal{U} \to \mathbb{R}$ in terms of $S^{ijk}{}_{pq}$ and $T^{\ell m}{}_r$. The answer is quite simple—and though we've chosen tensors of relatively low rank to simplify the notation, you can see what the answer for tensors of general type would be.

Now suppose $\hat{x} = (\hat{x}^1, \dots, \hat{x}^n) : \widehat{\mathcal{U}} \to \mathbb{R}^n$ is another coordinate chart on some open subset $\widehat{\mathcal{U}} \subset M$ such that $\mathcal{U} \cap \widehat{\mathcal{U}} \neq \emptyset$. Denote by $\widehat{T}^{i_1\dots i_k}{}_{j_1\dots j_\ell} : \widehat{\mathcal{U}} \to \mathbb{R}$ the component functions for a tensor field $T \in \Gamma(T^k_\ell M)$ in the new chart. As we mentioned in Problem Set 2, the basis vectors $\frac{\partial}{\partial x^j}$ and $\frac{\partial}{\partial \hat{x}^j}$ in T_pM for any point $p \in \mathcal{U} \cap \widehat{\mathcal{U}}$ are related to each other by

$$\frac{\partial}{\partial x^j} = \frac{\partial \hat{x}^i}{\partial x^j} \frac{\partial}{\partial \hat{x}^i}, \tag{1}$$

where this time we're using the summation convention to imply a summation over the repeated index i (it's considered a *lower* index in $\frac{\partial}{\partial \hat{x}^i}$ because it appears in the denominator). The partial derivatives $\frac{\partial \hat{x}^i}{\partial x^j}$ for each i and j should best be understood as smooth functions $\mathcal{U} \cap \widehat{\mathcal{U}} \to \mathbb{R}$, though of course we'd have to use the coordinates and express them as functions on the open set $x(\mathcal{U} \cap \widehat{\mathcal{U}}) \subset \mathbb{R}^n$ in order to compute them. Let us be more explicit: denote by $\frac{\partial}{\partial x^j}\big|_p$ the actual vector in T_pM which is the value of the coordinate vector field $\frac{\partial}{\partial x^j} \in \mathrm{Vec}(\mathcal{U})$ at $p \in \mathcal{U}$, and define $\frac{\partial}{\partial \hat{x}^j}\big|_p$ similarly for $p \in \widehat{\mathcal{U}}$. Then Equation (1) says that for all $p \in \mathcal{U} \cap \widehat{\mathcal{U}}$,

$$\left.\frac{\partial}{\partial x^j}\right|_p = \frac{\partial \hat{x}^i}{\partial x^j}(p) \left.\frac{\partial}{\partial \hat{x}^i}\right|_p.$$

(b) Derive a similar expression for the coordinate differentials dx^j in terms of $d\hat{x}^j$ and $\frac{\partial x^j}{\partial \hat{x}^i}$ at points in $\mathcal{U} \cap \widehat{\mathcal{U}}$. *Hint: Use the chain rule!*

(c) For a cotangent vector field $\lambda \in \Gamma(T^0_1)$ and a covariant rank 2 tensor field $T \in \Gamma(T^0_2 M)$, use the above relations between dx^j and $d\hat{x}^j$ to derive the transformation formulas

$$\hat{\lambda}_i = \lambda_j \frac{\partial x^j}{\partial \hat{x}^i} \quad \text{and} \quad \widehat{T}_{ij} = T_{k\ell} \frac{\partial x^k}{\partial \hat{x}^i} \frac{\partial x^\ell}{\partial \hat{x}^j},$$

relating the distinct sets of component functions over $\mathcal{U} \cap \widehat{\mathcal{U}}$.

(d) For a contravariant rank 2 tensor field $T \in \Gamma(T^2_0 M)$, derive

$$\widehat{T}^{ij} = \frac{\partial \hat{x}^i}{\partial x^k} \frac{\partial \hat{x}^j}{\partial x^\ell} T^{k\ell}.$$

(e) Finally, for a tensor field $A \in \Gamma(T^1_1 M)$ of "mixed" type $(1, 1)$, show that

$$\widehat{A}^i{}_j = \frac{\partial \hat{x}^i}{\partial x_k} A^k{}_\ell \frac{\partial x^\ell}{\partial \hat{x}^j}.$$

This formula has a nice interpretation using matrices: define the smooth matrix-valued function $\mathbf{A} : \mathcal{U} \to \mathbb{R}^{n \times n}$ *by setting the entry at the* i*th row and* j*th column of* $\mathbf{A}(p)$ *to* $A^i{}_j(p)$*, and define* $\widehat{\mathbf{A}} : \widehat{\mathcal{U}} \to \mathbb{R}^{n \times n}$ *similarly. We can also define the partial derivative matrix* $\mathbf{S} : \mathcal{U} \cap \widehat{\mathcal{U}} \to \mathbb{R}^{n \times n}$ *with entries* $\mathbf{S}^i{}_j = \frac{\partial \hat{x}^i}{\partial x^j}$*, and observe that by the inverse function theorem,* $\mathbf{S}^{-1}$ *is the matrix with entries* $\frac{\partial x^i}{\partial \hat{x}^j}$*. Then the transformation formula above becomes*

$$\widehat{\mathbf{A}} = \mathbf{S} \mathbf{A} \mathbf{S}^{-1}.$$