PROBLEM SET 8

Suggested reading

Lecture notes (on the website): Chapter 2, *Bundles* (up to §2.3); you might also find Chapter 1 informative, but it is inessential

Problems

1. In this problem, we prove that for any Lie group G with Lie algebra $\mathfrak{g} = T_e G$,

$$\exp(X+Y) = \exp(X)\exp(Y) \tag{1}$$

holds for any $X, Y \in \mathfrak{g}$ that satisfy $[X, Y] = 0$. Note that the formula clearly does not hold in general without this extra hypothesis, since $\exp(X)\exp(Y)$ would then always equal $\exp(Y)\exp(X)$ and thus imply that G is abelian near the identity. The latter turns out to be true however if $\mathfrak{g}$ is abelian; this fact was mentioned at the end of last Thursday's lecture, and we already proved the converse.

Throughout the following, we associate to any $X \in \mathfrak{g}$ the unique left-invariant vector field $X^L \in \mathrm{Vec}(G)$ with $X^L(e) = X$.

   (a) Recall that $\exp(tX) = \varphi^t_{X^L}(e)$ for every $t \in \mathbb{R}$ and $X \in \mathfrak{g}$. Use this characterization of the exponential map to show that for every $g \in G$, $X \in \mathfrak{g}$ and $t \in \mathbb{R}$,

$$g \exp(tX) = \varphi^t_{X^L}(g).$$

   In particular, this implies $\exp(sX)\exp(tY) = \varphi^t_{Y^L} \circ \varphi^s_{X^L}(e)$ for all $s, t \in \mathbb{R}$ and $X, Y \in \mathfrak{g}$.

   (b) Now assume $X, Y \in \mathfrak{g}$ satisfy $[X, Y] = 0$, and show that the paths $\alpha(t) := \exp(t(X+Y))$ and $\beta(t) := \exp(tX)\exp(tY)$ satisfy the same initial value problem, hence they are identical for all t, implying $\exp(X+Y) = \exp(X)\exp(Y)$.
   Hint: Recall that vector fields with vanishing Lie brackets have commuting flows.

   (c) One can show (cf. Proposition B.16 in the lecture notes) that every connected topological group is generated by a neighborhood of its identity element, hence the above result implies that a connected Lie group is abelian if and only if its Lie algebra is abelian. Find an example of a *disconnected* Lie group that is nonabelian even though its Lie algebra is abelian.
   Hint: All 1-dimensional Lie algebras are abelian. (Why?) Out of the simple examples of Lie groups that you know, is there one that is 1-dimensional and has multiple connected components?

2. We say that a smooth manifold M is *parallelizable* if its tangent bundle $TM \to M$ is trivial.

   (a) Show that every trivial vector bundle is orientable. (This is easy.) Hence every parallelizable manifold is orientable.

   (b) Show that every Lie group is parallelizable.
   Hint: Start by choosing an isomorphism of the tangent space at the identity to $\mathbb{R}^n$. Then think about left-invariant vector fields.

3. Given a Lie group G with Lie algebra $\mathfrak{g} = T_e G$ and an element $g \in G$, denote by $L_g, R_g : G \to G$ the diffeomorphisms $L_g(h) = gh$ and $R_g(h) = hg$; these maps are called *left translations* and *right translations* respectively. It will be useful to observe that while L_g and L_h do not generally commute with each other unless $gh = hg$, and a similar statement holds for right translations, every left translation commutes with every right translation. (Take a moment to convince yourself of this.)

A k-form ω on G is called *left-invariant* or *right-invariant* if it satisfies

$$L_g^*\omega = \omega \quad \text{or} \quad R_g^*\omega = \omega$$

respectively for all $g \in G$.

(a) Show that for every $\omega \in \Lambda^k \mathfrak{g}^*$ there exists a unique left-invariant k-form $\omega^L \in \Omega^k(G)$ and a unique right-invariant k-form $\omega^R \in \Omega^k(G)$ such that $\omega^L_e = \omega^R_e = \omega$. In particular, since $\dim \Lambda^n \mathfrak{g}^* = 1$ if $\dim G = n$, this means G admits left-invariant and right-invariant volume forms that are each unique up to multiplication by a constant.

(b) Show that if ω is a left-invariant k-form on G, then so is $R_g^*\omega$ for every $g \in G$.

(c) Denote $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$, regarded as a group with respect to multiplication. Parts (a) and (b) together imply the existence of a unique smooth function $f : G \to \mathbb{R}^*$ such that if μ is any left-invariant volume form on G,

$$R_g^*\mu = f(g)\mu.$$

Show that $f : G \to \mathbb{R}^*$ is a group homomorphism. We call f the *modular function* on G.

(d) A Lie group is called *unimodular* if its modular function is identically equal to 1, which means every left-invariant volume form is also right-invariant (in this case we say it is *bi-invariant*). Show that G is unimodular whenever (i) it is abelian, or (ii) it is compact and connected. *Hint: What can a compact and connected subgroup of* $\mathbb{R}^*$ *look like?*

(e) Out of the usual matrix groups $\mathrm{GL}(n, \mathbb{F})$, $\mathrm{SL}(n, \mathbb{F})$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$, which ones are (i) abelian, (ii) connected, (iii) compact? *Hint: Recall that all of them can be identified with subsets of* $\mathbb{F}^{n\times n} \cong \mathbb{F}^{n^2}$ *(here* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$*), so "compact" means the same thing as "closed and bounded".*

(f) Here is a simple example of a Lie group that is connected but neither abelian nor compact: the group of orientation-preserving *affine* transformations of $\mathbb{R}$,

$$\mathrm{Aff}_+(\mathbb{R}) := \left\{ \varphi \in \mathrm{Diff}(\mathbb{R}) \;\middle|\; \varphi(x) = ax + b \text{ for some } a > 0,\ b \in \mathbb{R} \right\}.$$

Compute the modular function on $\mathrm{Aff}_+(\mathbb{R})$, and check that it is a group homomorphism. *Hint: There is no need to actually compute a left-invariant volume form! Try to reduce the condition defining* $f(g)$ *to a relation between two elements of* $\Lambda^n \mathfrak{g}^*$*.*

Observe that we've proved the following rather nice result: every compact connected Lie group G *with a choice of orientation admits a unique bi-invariant volume form* μ *such that* $\int_G \mu = 1$*. Note that* μ *depends on a choice of orientation because the integral does, but only up to a sign.*

4. The simplest example of a closed, orientable but non-parallelizable (cf. Problem 2) manifold is S^2. To see this, we shall prove in this problem the so-called "hairy sphere" theorem: S^2 does not admit a continuous vector field that is nowhere zero. (In other words, "you can't comb the hair on a sphere".)

(a) Show that the normal bundle of $S^2 \subset \mathbb{R}^3$ is trivial.

(b) Show that if $M \subset \mathbb{R}^n$ is a smooth hypersurface (i.e. an $(n-1)$-dimensional submanifold) with trivial normal bundle, then M is orientable.

(c) Show that every parallelizable manifold admits a smooth vector field that is nowhere zero.

In preparation for proving the hairy sphere theorem, we now recall the notion of a *winding number*. A continuous function $f : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^2 \setminus \{0\}$ has winding number $\mathrm{wind}(f) = k \in \mathbb{Z}$ if, using polar coordinates (r, θ) on $\mathbb{R}^2$, f can be presented as

$$f([t]) = (r(t), \theta(t))$$

for some continuous functions $r, \theta : \mathbb{R} \to \mathbb{R}$ such that $\theta(t + 2\pi) = \theta(t) + 2\pi k$. The crucial property of $\mathrm{wind}(f)$ is that it is homotopy invariant: for any continuous family of functions $f_\tau : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^2 \setminus \{0\}$ depending on a parameter $\tau \in [0, 1]$, $\mathrm{wind}(f_0) = \mathrm{wind}(f_1)$.

(d) Recall (cf. Problem Set 5 #4) the standard *spherical coordinates* (θ, ϕ) on S^2, related to Cartesian coordinates (x, y, z) by $x = \sin\theta\cos\phi$, $y = \sin\theta\sin\phi$ and $z = \cos\theta$. Any continuous vector field

X on S^2 can then be written on the complement of the poles $p_+ := \{\theta = 0\}$ and $p_- := \{\theta = \pi\}$ in the form

$$X(\theta,\phi) = \Theta(\theta,\phi)\,\frac{\partial}{\partial\theta} + \Phi(\theta,\phi)\,\frac{\partial}{\partial\phi},$$

where Θ and Φ are continuous real-valued functions that are 2π-periodic in ϕ, hence these define a continuous family of functions

$$F_\theta := (\Theta(\theta,\cdot), \Phi(\theta,\cdot)) : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^2$$

for $\theta \in (0,\pi)$ such that $F_\theta([\phi]) = 0$ if and only if $X(\theta,\phi) = 0$. Show that if X is nonzero at both p_+ and p_-, then

$$\mathrm{wind}(F_\epsilon) = -1 \quad \text{and} \quad \mathrm{wind}(F_{1-\epsilon}) = 1$$

for all $\epsilon > 0$ sufficiently small. Deduce that X cannot then be nonzero everywhere.
Hint: First convince yourself that as long as ϵ is sufficiently small, the winding numbers in question cannot depend on your choice of X. Then choose X as conveniently as possible near the poles.

Combining the above results with Problem 2, we've proved that there exists no Lie group diffeomorphic to S^2, or put another way, one cannot define any group structure on S^2 for which the maps defined by the group operations are smooth.

5. Show that $\exp : \mathfrak{sl}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})$ is not surjective.
Hint: If $\mathbf{A} \in \mathfrak{sl}(2,\mathbb{R})$, what can you deduce about the eigenvalues of $e^{\mathbf{A}}$?

Remark: One can use techniques from Riemannian geometry to prove that $\exp : \mathfrak{g} \to G$ is surjective whenever the Lie group G is both connected and compact. We will see this explicitly in Problem 7 for the examples $\mathrm{SO}(2)$ and $\mathrm{SO}(3)$. Notice that $\mathrm{SL}(2,\mathbb{R})$ is connected (this follows easily from the fact that $\mathrm{GL}_+(2,\mathbb{R})$ is connected), but it is not compact.

6. In this problem we use exterior algebra to prove some basic properties of the cross product on $\mathbb{R}^3$, in particular that it satisfies the Jacobi identity and thus makes $\mathbb{R}^3$ into a Lie algebra. This is meant as preparation for Problem 7, though if you already feel you know everything that is important to know about the cross product, or if you just aren't in the mood for exterior algebra, feel free to skip this and move on to Problem 7.

Let $V = \mathbb{R}^n$, fix the standard basis $\mathbf{e}_1, \dots, \mathbf{e}_n \in V$, denote its dual basis by $\lambda^1, \dots, \lambda^n \in V^*$, and let $\langle\ ,\ \rangle$ denote the standard Euclidean inner product. The latter determines an isomorphism

$$V \to V^* : \mathbf{v} \mapsto \mathbf{v}_\flat := \langle \mathbf{v}, \cdot \rangle, \tag{2}$$

whose inverse we will denote by $V^* \to V : \alpha \mapsto \alpha^\sharp$. Notice that $(\mathbf{e}_i)_\flat = \lambda^i$ for each $i = 1, \dots, n$. Recall that for each $k = 0, \dots, n$, $\dim \Lambda^k V^* = \frac{n!}{k!(n-k)!}$, hence $\Lambda^k V^*$ and $\Lambda^{n-k} V^*$ are always isomorphic. In the following, we will define a specific isomorphism

$$* : \Lambda^k V^* \to \Lambda^{n-k} V^*,$$

called the *Hodge star* operator, which in the case $n = 3$, $k = 2$ is determined by

$$*(\lambda^1 \wedge \lambda^2) = \lambda^3, \quad *(\lambda^2 \wedge \lambda^3) = \lambda^1, \quad *(\lambda^3 \wedge \lambda^1) = \lambda^2. \tag{3}$$

We can then define a product on V^* by

$$\alpha \times \beta := *(\alpha \wedge \beta) \tag{4}$$

and use the so-called "musical" isomorphism (2) to transfer this definition to $V = \mathbb{R}^3$, i.e. by defining

$$\mathbf{v} \times \mathbf{w} := (\mathbf{v}_\flat \times \mathbf{w}_\flat)^\sharp.$$

Take a moment to convince yourself that this matches whatever definition of the cross product you've seen before. (It should suffice actually to observe that it is bilinear and antisymmetric, and to check how it behaves on basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.)

The right hand side of (4) makes sense for any n, but it is an $(n-2)$-form, which is not the same as a 1-form unless $n = 3$. This is one reason why the cross product operation is not defined in every dimension, but is special to dimension three. To define the Hodge star for arbitrary n, one starts by defining an inner product $\langle\ ,\ \rangle$ on each $\Lambda^k V^*$ such that the collection of all k-forms of the form $\lambda^{i_1} \wedge \ldots \wedge \lambda^{i_k}$ forms an orthonormal basis.[1] For $k = 1$, this means $\lambda^1, \ldots, \lambda^n$ is an orthonormal basis of V^*, so the musical isomorphisms are isometries, i.e. $\langle \mathbf{v}_\flat, \mathbf{w}_\flat \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$. For $k = n$, it means that the natural choice of volume form $\mu := \lambda^1 \wedge \ldots \wedge \lambda^n$ has unit norm. We then define $* : \Lambda^k V^* \to \Lambda^{n-k} V^*$ as the unique vector space isomorphism such that

$$\langle \alpha, \beta \rangle \, \mu = \alpha \wedge (*\beta) \quad \text{for all } \alpha, \beta \in \Lambda^k V^*. \tag{5}$$

To see that a map satisfying this relation exists and is unique, it suffices to observe that both sides of the relation are bilinear with respect to α and β, and then derive a formula for the case when α and β are both of the form $\lambda^{i_1} \wedge \ldots \wedge \lambda^{i_k}$.

(a) Verify that the general definition of $*$ given by (5) implies (3) for $n = 3$ and $k = 2$.

(b) Show that the inverse of $* : \Lambda^k V^* \to \Lambda^{n-k} V^*$ is $(-1)^{k(n-k)} * : \Lambda^{n-k} V^* \to \Lambda^k V^*$. In particular, there is no sign change when $n = 3$.

(c) Prove that the cross product on $\mathbb{R}^3$ satisfies

$$\langle \mathbf{u}, \mathbf{v} \times \mathbf{w} \rangle + \langle \mathbf{v} \times \mathbf{u}, \mathbf{w} \rangle = 0. \tag{6}$$

(d) Show that $\langle \alpha \wedge \beta, \gamma \rangle = \langle \beta, \iota_{\alpha^\sharp} \gamma \rangle$ for all $\alpha \in \Lambda^1 V^*$, $\beta \in \Lambda^{k-1} V^*$ and $\gamma \in \Lambda^k V^*$. In other words, the exterior product with α is the transpose of the interior product with $\alpha^\sharp$.
Hint: It suffices to check this for your favorite basis elements.

(e) Deduce the formula $\alpha \wedge *\beta = *(\iota_{\alpha^\sharp} \beta)$ and combine it with the graded Leibniz rule for the interior product to prove that the cross product satisfies

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{w} - \langle \mathbf{u}, \mathbf{w} \rangle \mathbf{v}. \tag{7}$$

(f) Use (7) to prove the Jacobi identity,

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = 0. \tag{8}$$

This proves that $(\mathbb{R}^3, \times)$ is a Lie algebra. We will see in the next problem which Lie algebra it is.

7. Let us clarify the reasons why SO(2) and SO(3) are regarded as groups of "rotations".

(a) Let $\mathbf{J}_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{so}(2)$. Show that every element of SO(2) can be written as $e^{\theta \mathbf{J}_0}$ for some $\theta \in \mathbb{R}$, and that this matrix acts on $\mathbb{R}^2$ by a rotation of angle θ.

(b) Define $\Phi : \mathbb{R}^3 \to \mathrm{End}(\mathbb{R}^3)$ in terms of the cross product on $\mathbb{R}^3$ by $\Phi(\mathbf{v})\mathbf{w} := \mathbf{v} \times \mathbf{w}$. Show that the image of this map is in $\mathfrak{so}(3)$, and that it defines a Lie algebra isomorphism from $(\mathbb{R}^3, \times)$ to $(\mathfrak{so}(3), [\ ,\ ])$, i.e. it is a vector space isomorphism $\mathbb{R}^3 \to \mathfrak{so}(3)$ satisfying

$$\Phi(\mathbf{v} \times \mathbf{w}) = [\Phi(\mathbf{v}), \Phi(\mathbf{w})] \quad \text{for all} \quad \mathbf{v}, \mathbf{w} \in \mathbb{R}^3.$$

Hint: There are at least two sensible ways you could go about this. One would be to just check what Φ does to your favorite basis of $\mathbb{R}^3$. If you prefer not choosing a basis, then you might find the relations (6) and (8) helpful.

(c) Show that every $\mathbf{A} \in \mathrm{SO}(3)$ has 1 as an eigenvalue, and that the dimension of the corresponding eigenspace is 1 unless $\mathbf{A} = \mathbb{1}$.

(d) Show that if $\mathbf{A} \in \mathrm{SO}(3)$ has a 1-dimensional eigenspace $\ell \subset \mathbb{R}^3$ with eigenvalue 1, then $\mathbf{A}$ acts as a rotation on the orthogonal complement of ℓ, and $\mathbf{A} = e^{\Phi(\mathbf{v})}$ for some $\mathbf{v} \in \ell$.

We can infer from this problem a new geometric interpretation of the cross product on $\mathbb{R}^3$: $\mathbf{v} \times \mathbf{w}$ measures the degree to which rotations about $\mathbf{v}$ fail to commute with rotations about $\mathbf{w}$.

[1] One can show that this inner product on $\Lambda^k V^*$ depends only on the inner product on V, not on the specific choice of orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_k$. As a consequence, $* : \Lambda^k V^* \to \Lambda^{n-k} V^*$ is well defined for any oriented n-dimensional vector space with an inner product, not just for the specific vector space $V = \mathbb{R}^n$ with its preferred basis.