Seminar announcement

General information

Instructor: Prof. Chris Wendl
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Office hour: Wednesdays 15:00–16:00

Seminar webpage: `http://www.mathematik.hu-berlin.de/~wendl/Winter2017/AtiyahSinger/`

Time and place: Thursdays 13:00–15:00 in 3.011 (Rudower Chaussee 25)

Language: The seminar can be run in German or in English (or a mixture), depending on the preferences of the participants.

Prerequisites: Contents of the HU's courses *Differentialgeometrie I*, *Topologie I*, and preferably also *Funktionalanalysis* (may be taken concurrently). Concurrent enrolment in *Topologie II* is recommended if you have not seen much algebraic topology before.

All participants will be assumed to be comfortable with smooth n-dimensional manifolds, differential forms and de Rham cohomology, Riemannian metrics, connections and curvature on vector bundles, plus essential notions from algebraic topology (fundamental group, covering spaces, singular homology), measure theory (the L^p-spaces), and linear functional analysis (bounded linear operators on Banach or Hilbert spaces, compact operators—at some point the spectral theorem for self-adjoint compact operators will be quoted, so you will have to take it on faith if you have not seen it before).

Overview

One of the most celebrated and influential mathematical developments of the 1960's, the Atiyah-Singer index theorem asserts the equality of two quantities, one of them fundamentally analytical in nature, and the other topological. The analytical quantity involves a certain class of linear partial differential equations in a geometric context: namely, for two smooth vector bundles E, F over a closed smooth manifold M, one considers linear partial differential operators $D : \Gamma(E) \to \Gamma(F)$. If such an operator is *elliptic*—a condition that can be read off from the coefficients of its highest-order term in local coordinates at each point—then one can use functional analysis to show that the linear map $D : \Gamma(E) \to \Gamma(F)$ is *Fredholm*, meaning its kernel is finite dimensional and its image has finite codimension. The *index* of D is then defined as

$$\operatorname{ind}(D) = \dim \ker D - \operatorname{codim} \operatorname{im} D \in \mathbb{Z}.$$

The reason to consider $\operatorname{ind}(D)$ instead of the seemingly more natural dimension of the solution space $\ker D = \{\eta \in \Gamma(E) \mid D\eta = 0\}$ is that the latter can change abruptly if the operator is perturbed or deformed, but the index cannot: under any continuous deformation of Fredholm operators, changes in $\dim \ker D$ are always canceled by changes in $\operatorname{codim} \operatorname{im} D$, so that $\operatorname{ind}(D)$ remains constant. This suggests that the index should depend on the topology of the setting rather than on the specifics of the operator D, and the Atiyah-Singer theorem makes this precise by expressing $\operatorname{ind}(D)$ in terms of characteristic (cohomology) classes determined by M and the bundles E and F.

To see why such a formula might be useful, here are a few results that can be viewed as special cases of it.

The Gauss-Bonnet formula

For a closed oriented surface Σ with a Riemannian metric and Gaussian curvature $K_G : \Sigma \to \mathbb{R}$, we have

$$\frac{1}{2\pi} \int_\Sigma K_G \, dA = \chi(\Sigma).$$

You might have been taught to think of the left hand side of this formula as geometric while the right hand side is topological, but here is a different perspective: using rudimentary Chern-Weil theory (to be discussed in the fourth week of this seminar), one can view the left hand side as the evaluation of the *first Chern class* $c_1(T\Sigma) \in H^2_{\mathrm{dR}}(\Sigma)$ of the vector bundle $T\Sigma \to \Sigma$ with its natural complex structure (determined by the conformal class of the metric), i.e. this is a *topological* quantity. For the right hand side, we borrow some ideas from Hodge theory and consider the first-order differential operator

$$d + d^* : \Gamma(\Lambda^0 T^*\Sigma \oplus \Lambda^2 T^*\Sigma) \to \Gamma(\Lambda^1 T^*\Sigma),$$

where $d^* : \Omega^2(\Sigma) \to \Omega^1(\Sigma)$ is the formal adjoint of the usual exterior derivative $d : \Omega^1(\Sigma) \to \Omega^2(\Sigma)$, defined via the relation $\langle d\lambda, \omega \rangle_{L^2} = \langle \lambda, d^*\omega \rangle_{L^2}$ in terms of the natural L^2-pairing on forms induced by the Riemannian metric. It is not hard to show that $\ker(d + d^*)$ is simply the direct sum of the space of constant functions $f : \Sigma \to \mathbb{R}$ (i.e. constant sections of $\Lambda^0 T^*\Sigma$) with the space of 2-forms ω satisfying $d^*\omega = 0$, which is the L^2-orthogonal complement of the space of exact 2-forms, hence

$$\ker(d + d^*) \cong H^0_{\mathrm{dR}}(\Sigma) \oplus H^2_{\mathrm{dR}}(\Sigma).$$

Similarly, $\mathrm{im}(d+d^*)$ turns out to be the L^2-orthogonal complement of a finite-dimensional subspace of $\Omega^1(\Sigma)$ that projects isomorphically to $H^1_{\mathrm{dR}}(\Sigma)$, giving the following analytical interpretation to the right hand side of the Gauss-Bonnet formula:

$$\mathrm{ind}(d + d^*) = \dim H^0_{\mathrm{dR}}(\Sigma) + \dim H^2_{\mathrm{dR}}(\Sigma) - \dim H^1_{\mathrm{dR}}(\Sigma) = \chi(\Sigma).$$

In other words, Gauss-Bonnet expresses the index of an elliptic operator in terms of a characteristic class of a vector bundle:

$$\mathrm{ind}(d + d^*) = \langle c_1(T\Sigma), [\Sigma] \rangle := \int_\Sigma c_1(T\Sigma).$$

The Hirzebruch signature theorem

If M is a closed oriented manifold of dimension n, then for any $k \in \{0, \ldots, n\}$, the wedge product descends to de Rham cohomology as a nondegenerate pairing

$$H^k_{\mathrm{dR}}(M) \times H^{n-k}_{\mathrm{dR}}(M) \to \mathbb{R} : ([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta.$$

Whenever $n = 4m$ for some $m \in \mathbb{N}$, we can take $k := 2m$ and view this pairing as a nondegenerate quadratic form on $H^{2m}_{\mathrm{dR}}(M)$; it is equivalent (under Poincaré duality) to the *intersection form* of M, which is defined on the homology $H_{2m}(M)$. As with all nondegenerate quadratic forms, the maximal dimensions $b_\pm \geq 0$ of subspaces on which the form is positive- or negative-definite respectively satisfy $b_+ + b_- = \dim H^{2m}_{\mathrm{dR}}(M)$, and the difference between these two integers is called the *signature* of M,

$$\sigma(M) := b_+ - b_- \in \mathbb{Z}.$$

Similarly to the Euler characteristic in our discussion of Gauss-Bonnet, the signature $\sigma(M)$ can be expressed as the index of an elliptic operator, and the resulting special case of the Atiyah-Singer formula (actually proved slightly earlier by Hirzebruch) expresses this index in terms of the so-called *Pontrjagin classes* $p_k(M) \in H^{4k}(M)$ for $k \in \mathbb{N}$, e.g. if $\dim M = 4$, we obtain

$$\sigma(M) = \frac{1}{3} \int_M p_1(M).$$

One application of this formula is to obstruct the existence of almost complex or symplectic structures on certain 4-manifolds. By an elementary result in symplectic geometry, if M has a symplectic form ω, then it also admits an almost complex structure, i.e. its tangent bundle TM can be made into a complex vector bundle. In this case, one can relate the first Pontrjagin class to the first Chern class $c_1(TM) \in H^2(M)$ and the Euler characteristic $\chi(M)$, transforming the Hirzebruch formula into the relation

$$2\chi(M) + 3\sigma(M) = \int_M c_1(TM) \wedge c_1(TM).$$

Note that the left hand side of this formula is purely topological, while the right side depends on the almost complex structure (because the first Chern class does). But if M is, for example, the connected sum $\mathbb{C}P^2 \# \mathbb{C}P^2$, then it is an easy exercise in homology to show that no choice of cohomology class $c_1(TM) \in H^2(\mathbb{C}P^2 \# \mathbb{C}P^2)$ can ever make both sides of this relation equal: it follows that $\mathbb{C}P^2 \# \mathbb{C}P^2$ admits no almost complex structure, and therefore also no symplectic form.

Moduli spaces

While the above applications were already understood when the general index theorem was discovered, another important class of applications became clear around 15 years later. In differential topology, one way of defining invariants that are sensitive to the smooth structure of a manifold is by studying the solution spaces of nonlinear PDEs. The first examples of such PDEs came from gauge-theoretic physics and arose in Donaldson's pioneering work on smooth 4-manifolds in the early 1980's; later examples include the Seiberg-Witten invariants of smooth 4-manifolds, and the theory of pseudoholomorphic curves in symplectic manifolds. The main idea is that for *elliptic* PDEs, solution spaces tend to have a nice structure, e.g. in the best cases they form compact smooth finite-dimensional manifolds ("moduli spaces"), and the topology of these moduli spaces can be used to distinguish manifolds with different smooth or symplectic structures. To make this idea work, one needs to be able to compute the dimensions of such moduli spaces, and this is also an index problem: each tangent space to a well-behaved moduli space can be viewed as the kernel of a linear elliptic operator that is surjective, so its index is precisely the dimension. In this way, the Atiyah-Singer formula and its various corollaries have become essential tools in the development of smooth or symplectic invariants based on nonlinear elliptic PDEs.

Objective

While the general version of the index theorem applies to all linear elliptic operators, in this seminar we will consider only the special class of so-called *Dirac operators*. The motivation for this restriction is three-fold:

1. All of the applications mentioned above can be derived from special cases of Dirac operators.
2. One can use topological arguments to deduce the general case from the index theorem for Dirac operators (though we will not do that in this semester).
3. The theorem for Dirac operators can be proved without needing to develop too much abstract topological machinery.

The proof we will work through is based on the "heat kernel" approach: it is not the original proof of Atiyah and Singer, but was developed slightly later and involves standard methods from functional analysis.

Literature

The seminar will mostly follow:

- John Roe, *Elliptic Operators, Topology and Asymptotic Methods* (2nd edition), Chapman & Hall 1998

This is probably the most digestible book dealing with the heat kernel proof of the index theorem for Dirac operators, though it is not the only one. For a slightly different (and sometimes more sophisticated) perspective on spin geometry and Dirac operators, you may sometimes find it helpful to consult either of the following:

- Nicole Berline, Ezra Getzler and Michèle Vergne, *Heat kernels and Dirac operators*, Springer 2004
- H. Blaine Lawson and Marie-Louise Michelsohn, *Spin Geometry*, Princeton University Press 1989

The following book is also worth mentioning, though it approaches the subject from a very different perspective than that of our seminar:

- B. Booss and D. D. Bleecker, *Topology and Analysis: The Atiyah-Singer Index Formula and Gauge-Theoretic Physics*, Springer Universitext 1985

Actually, this is mostly just a translation of

- B. Booß, *Topologie und Analysis: Eine Einführung in die Atiyah-Singer Indexformel*, Springer 1976

Requirements

All (students and otherwise) are welcome to attend the seminar and may volunteer to give talks. For students to receive credit, the requirements are the following:

1. Give at least one of the talks (normally this will mean one 90-minute session), with careful attention to fitting all necessary material within the given time constraints;
2. Submit clearly readable notes for your talk (handwritten is fine), so that they can be scanned and uploaded to the website by the following day;
3. Attend the seminar regularly (at most two absences in the semester).

The schedule of talks and assignment of topics will be decided in the first week of the semester (see the tentative plan below). **Participants are asked to send an e-mail to `wendl@math.hu-berlin.de` by the end of Friday, October 20**, telling me the following:

- Confirmation that you would like to give a talk;
- Any information about your background that might be helpful in assigning topics, e.g. if you are more confortable with algebra or topology than analysis, or vice versa, or if you would like to request a specific topic;
- In case you would be interested in dividing up a topic with someone else, tell me who (there are at least two topics that will likely need to be split over two sessions).

Tentative plan of topics

Here is a list of topics corresponding to the schedule posted on the website. I plan to give the first three talks myself since they will adhere less closely to Roe's book, and probably also the last one, which may be omitted if we end up running short of time.

1. *General introduction*
   Some examples of index formulas, motivation for Dirac operators, idea of the heat kernel proof.

2. *Sobolev spaces and partial differential operators on manifolds*
   Crash course on Sobolev spaces defined via Fourier series/transforms, then generalized to spaces of sections of vector bundles on closed manifolds, including the Sobolev embedding theorem and Rellich compactness theorem.

3. *Ellipticity, regularity, Fredholm and spectral properties*
   What it means for an operator to be elliptic, why (weak) solutions to elliptic equations are smooth, sketch of the proof that elliptic operators are Fredholm, and properties of their spectrum in the self-adjoint case.

4. *Characteristic classes and Chern-Weil theory (Roe, chapter 2)*
   Using connections and curvature to define characteristic classes of vector bundles $E \to M$ as elements of the de Rham cohomology $H^*_{\mathrm{dR}}(M)$, including the Chern character, the $\hat{\mathcal{A}}$-genus and Hirzebruch's $\mathcal{L}$-genus.

5. *Clifford algebras and Dirac operators (Roe, chapter 3)*
   Introducing the notions of Clifford bundles and Dirac operators, some basic examples, and the Weitzenböck formula relating the square of a Dirac operator to the Laplacian.

6. *Spin groups (Roe, chapter 4)—probably requires two sessions*
   Definitions of the groups $\mathrm{Pin}(k)$ and $\mathrm{Spin}(k)$, the double cover $\mathrm{Spin}(k) \to \mathrm{SO}(k)$, representation theory of Clifford algebras, spin structures on manifolds.

7. *The heat and wave equations (Roe, chapter 7)*
   How to solve the heat equation for a Dirac operator, definition of the heat kernel and its approximation via local data.

8. *Traces and eigenvalue asymptotics (Roe, chapter 8)—requires half a session*
   The notions of Hilbert-Schmidt and trace-class operators, Weyl's asymptotic formula for eigenvalues of the Laplacian.

9. *The quantum harmonic oscillator (Roe, first section of chapter 9)—requires half a session*
   Mehler's formula for the harmonic oscillator heat kernel.

10. *The index problem for graded Dirac operators (Roe, chapter 11)*
    Expressing the index of a graded Dirac operator as an integral of an asymptotic expansion coefficient.

11. *The Getzler calculus and the local index theorem (Roe, chapter 12)—probably requires two sessions*
    This talk completes the proof of the index theorem by relating the aforementioned asymptotic expansion coefficient to the $\hat{\mathcal{A}}$-genus and Chern character.

12. *Applications of the index theorem (Roe, chapter 13)*
    Proofs of the Hirzebruch signature theorem and Hirzebruch-Riemann-Roch theorem, and whatever else there is time for.

13. *Witten's approach to Morse theory (Roe, chapter 14)*
    How to deduce the Morse inequalities from a deformation of the de Rham complex (just for fun).