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The harmonic oscillator

(First section of Chapter 9 in Roe's book)
All the proofs are straightforward, so not included here.

We fix a positive real number a.

Def: On $L^2(\mathbb{R})$, we define

- the harmonic oscillator: $H := \frac{-d^2}{dx^2} + a^2x^2$,
- the annihilation operator: $A := ax + \frac{d}{dx}$,
- the creation operator: $A^* := ax - \frac{d}{dx}$.

Properties: • $H, A, A^* : S(\mathbb{R}) \circlearrowleft$ (Schwartz space).

- $AA^* = H + a$,
- $A^*A = H - a$,

$\Rightarrow$ • $[A, A^*] = 2a$,

- $[H, A] = -2aA$,
- $[H, A^*] = 2aA^*$.

Definition: • The "ground state" of H is $\psi_0 \in L^2(\mathbb{R})$ satisfying that $\|\psi_0\| = 1$ and $A\psi_0 = 0$.

For each $k \geq 1$ integer, the "excited state" of H is defined inductively by)

$$\psi_k := \frac{1}{(2ka)^{1/2}} A^* \psi_{k-1}.$$

Lemma: (i) $\psi_0 = a^{1/2}\pi^{1/4} e^{-\frac{ax^2}{2}}$

(ii) $\psi_k \in S(\mathbb{R})$, $\|\psi_k\| = 1$ and $H\psi_k = (2k+1)a\,\psi_k$.

(iii) $\psi_k(x) = h_k(x)\, e^{-\frac{ax^2}{2}}$ with h_k a polynomial of degree k with positive leading coeff (which coincides up to a constant with a Hermite polynomial).

(iv) $\mathrm{span}_{\mathbb{R}}\{\psi_k\}_{k\geq 0} = \mathcal{P} := \{x \mapsto p(x) e^{-\frac{x^2 a}{2}} : p \text{ polynomial}\}$ is dense in $L^2(\mathbb{R})$.

Proof: (i) is easy to check, but Roe finds it.

(ii) follows using the properties.

(iii) follows by the definition of ψ_k and the recurrence relation satisfied by Hermite polynomials.

(iv) is equivalent to the classical fact that Hermite polynomials form an orthomormal basis of the space $L^2(\mathbb{R}, \omega(x)dx)$ where dx is the canonical volume element in $\mathbb{R}$ and the weigth $\omega(x)$ is given by e^{-ax^2}, that is, $\|f\|^2 = \int_{\mathbb{R}} |f(x)|^2 \omega(x)\,dx$. ■

Conclusion: $L^2(\mathbb{R})$ admits a complete orthogonal decomposition into (1-dimensional) eigenspaces for H, with discrete spectrum tending to infinity.

By replacing Δ by H in the heat equation we obtain the "harmonic oscilator heat equation"

$$\frac{\partial u}{\partial t} + H u = 0.$$

The next result, called Mehler's formula, is ~~detaided~~ demonstrated in Roe's book.

Proposition: The harmonic oscillator heat kernel satisfies

$$u(x,t) = \sqrt{\frac{a}{2\pi \sinh(2at)}} \exp\left(\frac{-ax^2 \coth(2at)}{2}\right).$$