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Applications of the index theorem

Seminar: The Atiyah-Singer Index theorem

Applications of the Atiyah-Singer Index Theorem

- The Spinor Dirac operator:

Let M be compact, 2m-dim spin-manifold, Δ the assoc. spin bundle, D the Dirac operator on Δ. The index thm then has the form:

$$\operatorname{Ind}(D) = \int_M \hat{A}(M)$$

Recall that the $\hat{A}$-genus was given as the pontryagin-genus associated to $z \mapsto \frac{\sqrt{z}/2}{\sinh \sqrt{z}/2}$.

A while ago we showed the following statements

-) In general: $D^2 = \nabla^* \nabla + F^S + \frac{1}{4} K$

-) For the spin bundle, $F^S = 0$.

-) (Bochner vanishing argument) If the least eigenvalue of K at each point of M is strictly positive, then $\ker D^2 = 0$

From this we conclude

Theorem Let M be compact mfld which admits a spin structure. If $\hat{A}$ genus $\langle \hat{A}(M), [M] \rangle$ is non-zero, then M admits no metric of strictly positive scalar curvature K.

Proof We have $D^2 = \nabla^* \nabla + \frac{1}{4} K$. If $K > 0 \Rightarrow \ker D = \ker D^2$ is zero. But then $\operatorname{Ind} D = \langle \hat{A}(M), [M] \rangle = 0$, a contradiction. ▨

Remark: In dimension 4, we have an isomorphism of algebras $Cl(\mathbb{R}^4) \to M_2(\mathbb{H})$ defined by

$$e_1 \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad e_2 \mapsto \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} \quad e_3 \mapsto \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} \quad e_4 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

$\Rightarrow$ $Cl(\mathbb{R}^4)$ has quaternionic structure and acts on $\mathbb{H}^2$, and $Cl(\mathbb{R}^4)$ acts on $\mathbb{H}^2 \cong \mathbb{C}^4$, which is just the spin-representation

$\Rightarrow$ Spin(4) also has quaternionic structure.

We can use this to get

<u>Proposition</u> The $\hat{A}$-genus of a 4-dim spin mfld is an <u>even</u> integer

<u>Proof</u>: The spinor bundle Δ can be considered as a bundle of quarternionic vector spaces by the above argument, and since this quart. structure is compatible with ∇ and commutes with Clifford action, $\ker D_+$ and $\ker D_-$ are quarternionic V.spaces too.

$\Rightarrow \operatorname{Ind}(D) = \int_M \hat{A}(M)$ is even. $\square$

- <u>The signature theorem</u>

Again, let M be smooth, $2m$-dim, compact, and <u>oriented</u>.

Reminder: we can consider $\Lambda^* T^*M \otimes \mathbb{C}$ as a Clifford bundle, via the isomorphism (of real vector bundles) $C\ell(TM) \xrightarrow{\sim} \Lambda^* TM \xrightarrow{\sim} \Lambda^* T^*M$.

The dirac operator D is the given by the De Rahm operator $d + d^*$.

Some theorems from Hodge theory:

<u>Theorem</u> (Hodge) $H^n(M, \mathbb{C}) \cong \mathcal{H}_n := \ker (d+d^*)^2 \subset C^\infty(\Lambda^n T^*M)$

<u>Theorem</u> (Poincare Duality) The cap product

$$H^k(M; \mathbb{C}) \otimes H^{n-k}(M; \mathbb{C}) \to H^n(M, \mathbb{C}) \xrightarrow[\cong]{\int} \mathbb{C}$$

is a non-degenerate pairing.

We equip $\Lambda^* T^*M \otimes \mathbb{C}$ with the canonical grading given by the Clifford action of $i^m \omega$. On p-forms this just $\varepsilon = i^m \omega = i^{m+p(p-1)} *$.

Call D with this grading the signature operator of M.

Suppose m is even, i.e. $\dim M$ is a multiple of 4.

Then $H^m(M; \mathbb{C}) \otimes H^m(M; \mathbb{C}) \to \mathbb{C}$ is symmetric bilinear.

<u>Def</u>: The signature of M is the signature of this form (called intersection form)

<u>Prop</u>: $\operatorname{Sign}(M) = \operatorname{Ind}(D)$.

<u>Proof</u>: Write $\Delta = D^2$ as $\Delta^+ \oplus \Delta^-$ with respect to grading ε.

Then $\mathrm{Ind}(D) = \dim\ker(\Delta^+) - \dim\ker(\Delta^-)$

Split Δ^+, Δ^- further into $\Delta^+_\ell, \Delta^-_\ell$, with respect to the ε-inv. subspaces $C^\infty(\Lambda^\ell T^*M \oplus \Lambda^{2m-\ell}T^*M)$ for $0 \le \ell < m$ and $C^\infty(\Lambda^m T^*M)$ for $\ell = m$.

<u>Claim</u> if $\ell < m$, then $\ker(\Delta^+_\ell) \cong \ker(\Delta^-_\ell)$.

Suppose $\alpha = \underset{\ell\text{-form}}{\beta} + \underset{2m-\ell\text{ form}}{\gamma} \in \ker(\Delta^+_-)$, with β, γ harmonic. Since α ε-invariant, we must have $\gamma = \varepsilon(\beta)$. We then have the map

$$\begin{cases} \ker(\Delta^+_\ell) \longrightarrow \ker(\Delta^-_\ell) \\ \beta + \varepsilon(\beta) \longmapsto \beta - \varepsilon(\beta) \end{cases}$$

which is clearly an iso.

We conclude that $\mathrm{Ind}(D) = \dim \underbrace{\ker(\Delta^+_m)}_{\mathcal{H}^+_m} - \dim \underbrace{\ker(\Delta^-_m)}_{\mathcal{H}^-_m}$

But $\mathcal{H}^+_m$ and $\mathcal{H}^-_m$ are just the ± 1 eigenspaces of $*$ acting on $\mathcal{H}_m$, and the intersection form is given by $\alpha \mapsto \int \alpha \wedge \alpha$ which is pos. definite on $\mathcal{H}^+_m$, neg. def on $\mathcal{H}^-_m$, hence

$\mathrm{Ind}(D) = \mathrm{Sign}(M)$. $\boxtimes$

We need 2 Lemmas.

<u>Lemma</u> Let M be Spin, Δ associated spin bundle. Then $ch(\Delta) = 2^m\,\mathfrak{S}(TM)$, with $\mathfrak{S}$ pontryagin genus to $z \mapsto \cosh(\frac{\sqrt{z}}{2})$.

<u>Lemma</u> $ch(S/\Delta) = 2^m\mathfrak{S}(TM)$ for $S = \Lambda^* T^*M$.

<u>Proof</u>: $S \cong Cl(TM)$ with canonical grading

Locally $Cl = \Delta \otimes \Delta^* \Rightarrow$ locally $ch(S/\Delta) = ch(\Delta^*) \underset{\text{prev. lemma}}{=} 2^m\mathfrak{S}(TM)$

Since these calculations are local, they remain valid even if M has no global spin structure.

Recall: $\mathcal{L}$ class was pontryagin-genus to $z \mapsto \frac{\sqrt{z}}{\tanh(\sqrt{z})}$

Theorem (Hirzebruch Signature theorem) If $\dim M$ divisible by 4

$$\mathrm{Sign}(M) = \langle \mathcal{L}(TM), [M] \rangle$$

Proof: We have from before

$$\mathrm{Sign}(M) = \mathrm{Ind}(D) = \langle \hat{A}(TM) \cdot ch(S/\Delta), [M] \rangle \overset{\text{Lemma}}{=} \langle 2^m \underbrace{\hat{A}(TM)}_{\text{assoc } \frac{\sqrt{z}/2}{\sinh(\sqrt{z}/2)}} \underbrace{G(TM)}_{\cosh(\sqrt{z}/2)}, [M] \rangle$$

So $\mathcal{L}_1(TM) := \hat{A}(TM) \cdot G(TM)$ is assoc. to

$$g_1(z) = \frac{\sqrt{z}/2}{\sinh(\sqrt{z}/2)} \cosh\left(\frac{z}{2}\right) = \frac{\sqrt{z}/2}{\tanh(\sqrt{z}/2)}$$

Compare $\mathcal{L}(TM)$ assoc. to $\frac{\sqrt{z}}{\tanh(\sqrt{z})}$, so

$$|\mathcal{L}_1(TM)|_k = 2^{-k/2} |\mathcal{L}(TM)|_k \quad \text{so} \quad 2^m |\mathcal{L}_1(TM)|_{2m} = |\mathcal{L}(TM)|_{2m} \qquad \blacksquare$$

Remark: We can compute the first couple terms of $\hat{A}$ and $\hat{\mathcal{L}}$

$$\hat{A} = 1 - \frac{p_1}{24} + \frac{(-4p_2 + 7p_1^2)}{5760} + \dots$$

$$\hat{\mathcal{L}} = 1 + \underbrace{\frac{p_1}{3}}_{\deg 4} + \underbrace{\frac{(7p_2 - p_1^2)}{45}}_{\deg 8} + \dots$$

so in part. for $\dim M = 4$ $\hat{\mathcal{L}}$-genus $= -8 \cdot \hat{A}$-genus

We get

Theorem (Rochlin) The signature of a (smooth) spin 4-fold is divisible by 16.

Remark: The signature can be defined for a topological spin 4-fold too. However the above theorem is not true then, there are examples of compact topological spin 4-folds with signature only divisible by 8. (Freedman)

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Applications

- The Hirzebruch-Riemann-Roch Theorem

Some linear algebra: If V real ~~vectorspace~~ inner product space with complex structure $J \in V$, $J^2 = -1$ then

$$V \otimes_{\mathbb{R}} \mathbb{C} = V^+ \oplus V^-, \text{ given by the projection } p^{\pm} = \frac{1}{2}(1 \mp iJ)$$

these are the $\pm i$ Eigenspaces of $J \otimes id$ and we have

$$V \cong V^+ \cong \overline{V^-} \text{ as complex vectorspaces.}$$

~~Now look at M compact n-dimensional complex mfld.~~

We can make $\Lambda^* V^+$ into a $Cl(V) \otimes \mathbb{C}$ representation by

for $p \in V^+$, $q \in V^-$, $x \in \Lambda^* V^+$, define $(p+q) \cdot x = \sqrt{2}(p \wedge x + q \lrcorner x)$

this satisfies $p^2 = q^2 = 0$, $pq + qp = -2(p,q)$.

Note: This repr. has, if $\dim V = 2m$, $\dim = 2^m$, compared with the regular representation, which has 2^{2m}, and is called spin representation of $Cl(V) \otimes \mathbb{C}$.

Now let M be compact n-dimensional complex mfld.

We then get $TM \otimes \mathbb{C} \cong T^{1,0}M \oplus T^{0,1}M$, and $S = \Lambda^*(T^{0,1}M)^*$ carries a spin representation of $Cl(TM)$.

Now let W be a holomorphic vector bundle over M.

We can form the Dolbeault complex

$$\Omega^{0,0}(W) \xrightarrow{\bar\partial} \Omega^{0,1}(W) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \Omega^{0,n}(W)$$

with $\Omega^{0,k}(W) =$ (smooth) sections of $\Lambda^k (T^{0,1}M)^* \otimes W$, i.e. sections inside $S \otimes W$.

By Hodge theory we have that

$$\chi(W) = \cancel{\sum (-1)^k \dim \Omega^{0,k}(W)} = \sum (-1)^k \dim H^{0,k} W = \operatorname{Ind}(\bar\partial + \bar\partial^*)$$

with $H^{0,k}(W)$ defined by homology of this complex.

Note $H^{0,0}(W) =$ holomorphic sections of W

Now, we have the prop.

Prop: If M is Kahler, then $D = \sqrt{2}(\bar\partial + \bar\partial^*)$. In general
$D = \sqrt{2}(\bar\partial + \bar\partial^*) + A$, with A endomorphism of S.

We then have the homotopy $D - tA$, $t \in [0,1]$,
which gives $\mathrm{Ind}(D) = \mathrm{Ind}(\bar\partial + \bar\partial^*) = \chi(W)$.

Let us state the main theorem.

Theorem (Hirzebruch-Riemann-Roch).
We have $\chi(W) = \langle Td(T^{1,0}M)\, ch(W), [M] \rangle$
where Td is the Chern genus assoc. to $z \mapsto \frac{z}{e^z - 1}$.

Sketch of Proof: We already have $\chi(W) = \mathrm{Ind}(D) = \langle \hat{A}(TM)\, ch(S/\Delta)\, ch(W), [M] \rangle$.
We have (without proof)

Lemma: $ch(S/\Delta)$ = Chern genus of $T^{1,0}M$ associated to $z \mapsto e^{-z/2}$

Lemma: $\hat{A}(TM)$ = Chern genus of $TM \otimes \mathbb{C}$ assoc. to $z \mapsto \frac{z/2}{\sinh z/2}$

$\Rightarrow$ $\hat{A}(TM)\, ch(S/\Delta)$ Chern genus assoc. to $\frac{z/2}{\sinh z/2} \cdot e^{-z/2} = \frac{z}{e^z - 1}$ ∎