SEMINAR: ATIYAH-SINGER INDEX THEOREM (WINTERSEMESTER 2017-18)

OVERVIEW

Fix smooth n-mfd M & smooth cpx vector bndls $E, F \longrightarrow M$ of ranks $\ell, m \in \mathbb{N}$.

defn: A linear partial differential operator from E to F of order $k \in \mathbb{N}$ is a linear map $D : \Gamma(E) \longrightarrow \Gamma(F)$ s.t. for any chart $M \supseteq \mathcal{U} \xrightarrow{(x^1, \dots, x^n)} \mathbb{R}^n$ & local trivs. of E & F over $\mathcal{U}$, D acts on sections over $\mathcal{U}$ as $C^\infty(\mathcal{U}, \mathbb{C}^\ell) \longrightarrow C^\infty(\mathcal{U}, \mathbb{C}^m)$,

$$(D\eta)(x) = \sum_{|\alpha| \le k} c_\alpha(x) \partial^\alpha \eta(x) \quad \text{for some smooth fns. } c_\alpha : \mathcal{U} \longrightarrow \mathbb{C}^{m \times \ell}.$$

(Multi-index notation: $\alpha = (\alpha_1, \dots, \alpha_n)$ for integers $\alpha_j \ge 0$, so $\partial^\alpha := \left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x^n}\right)^{\alpha_n}$ has order $|\alpha| := \sum_{j=1}^n \alpha_j$.)

exs: (1) $\underbrace{E = \Lambda^m T^*M \otimes \mathbb{C}}_{\text{complexification of } \Lambda^m T^*M}$, $F = \Lambda^{m+1} T^*M \otimes \mathbb{C}$, $d : \underset{\Omega^m(M, \mathbb{C})}{\underset{\shortparallel}{\Gamma(E)}} \longrightarrow \underset{\Omega^{m+1}(M, \mathbb{C})}{\underset{\shortparallel}{\Gamma(F)}}$ has order 1.

(Remark: omit complexifications from now on.)

(2) For M a cpx. mfd. & $E \to M$ a holomorphic vec. bndl. (i.e. all transition maps are hol.), $\exists$ natural Cauchy-Riemann operator

$$\bar\partial : \Gamma(E) \longrightarrow \Gamma(\underbrace{\overline{\mathrm{Hom}}_{\mathbb{C}}(TM, E)}_{\mathbb{C}\text{-antilinear maps } TM \to E}) =: \Omega^{0,1}(M, E),$$

defined in any hol. local triv. of E as $C^\infty(\mathcal{U}, \mathbb{C}^\ell) \longrightarrow \Omega^{0,1}(\mathcal{U}, \mathbb{C}^\ell)$ by $\bar\partial\eta :=$ $\mathbb{C}$-antilinear part of $d\eta = \frac{1}{2}(d\eta + i \circ d\eta \circ i)$.

Order $= 1$, $\ker \bar\partial = \{$hol. sections of $E\}$.

(3) (M, g) Riem. mfd. $\rightsquigarrow$ induced bndl metric $\langle \,, \rangle$ on $\Lambda^m T^*M$ & volume element vol_g $\rightsquigarrow$ L^2-pairing on $\Omega^m(M)$: $\langle \alpha, \beta \rangle_{L^2} := \int_M \langle \alpha, \beta \rangle \, \mathrm{vol}_g$.

Then $d : \Omega^m(M) \longrightarrow \Omega^{m+1}(M)$ has a formal adjoint $d^* : \Omega^{m+1}(M) \longrightarrow \Omega^m(M)$ s.t. $\langle \alpha, d\beta \rangle_{L^2} = \langle d^*\alpha, \beta \rangle_{L^2}$ $\forall \alpha \in \Omega^{m+1}(M)$, $\beta \in \Omega^m(M)$ w/ cpct support (integration by parts = Stokes' thm)

d^* is 1st order.

(4) Hodge Laplacian on (M, g): $\Delta := dd^* + d^*d = (d + d^*)^2 : \Omega^m(M) \circlearrowleft$

2nd order, e.g. on $(\mathbb{R}^n, g_{\mathrm{Eucl}})$, $m = 0 \Rightarrow \Delta = d^*d : C^\infty(M) \circlearrowleft$, then $df = \partial_i f \, dx^i$, $d^*(\lambda_i dx^i) = -\sum_i \partial_i \lambda_i \Rightarrow \Delta f = -\sum_i \partial_i \partial_i f$.

thm: If M is closed & D is elliptic (to be def'd in week 3), then it is Fredholm, i.e. $\dim \ker D < \infty$ & $\mathrm{im}\, D \subseteq \Gamma(F)$ is a closed subspace of finite codimension.

defn: $\mathrm{ind}(D) := \dim\ker D - \mathrm{codim}\ \mathrm{im}\, D \in \mathbb{Z}$.

ex: d & d^* not elliptic, e.g. $\dim\{\lambda \in \Omega^1(M) \mid d\lambda = 0\} = \infty$.

$\bar\partial$ & Δ are elliptic. So is $d + d^*$ from $\bigoplus_{k\text{ even}} \Lambda^k T^*M$ to $\bigoplus_{k\text{ odd}} \Lambda^k T^*M$

(square root of Δ!)

bad / hard Q: What is $\dim\ker D$?

trouble: Answer can change abruptly if D is perturbed.

better Q: What is $\mathrm{ind}(D)$?

prop: Given Banach spaces X, Y, $\mathrm{Fred}(X,Y) := \{\text{Fredholm operators } X \to Y\}$ is an open subset of the space of all bdd linear maps $X \to Y$, & $\mathrm{ind} : \mathrm{Fred}(X,Y) \to \mathbb{Z}$ is locally constant.

ex: $T : \mathbb{C}^m \to \mathbb{C}^n$ always has $\mathrm{ind}(T) = m - n$, independent of T.

MAIN PROBLEM: Compute $\mathrm{ind}(D)$ in terms of topological invts of M, E & F.

ex 1: (Σ, g) a closed oriented Riem. 2-mfd, $d + d^* : \Omega^0(\Sigma) \oplus \Omega^2(\Sigma) \to \Omega^1(\Sigma)$.

$\ker(d + d^*) = \{(f, \omega) \in C^\infty(\Sigma) \oplus \Omega^2(\Sigma) \mid df = 0 \text{ \& } d^*\omega = 0\}$

$\Leftrightarrow \omega \in (d(\Omega^1(\Sigma)))^\perp$ (L^2-orthogonal complement in $\Omega^2(\Sigma)$)

$\Rightarrow \dim\ker(d + d^*) = \dim H^0_{dR}(\Sigma) + \dim H^2_{dR}(\Sigma)$.

Similarly (using ideas from Hodge theory), $\mathrm{im}(d + d^*) = \{df + d^*\omega \mid f \in C^\infty(\Sigma),\ \omega \in \Omega^2(\Sigma)\}$

$= L^2$-ortho. complement in $\Omega^1(\Sigma)$ of a fin.-dim. subspace $\cong H^1_{dR}(\Sigma)$

$\therefore \mathrm{ind}(d + d^*) = \sum_k (-1)^k \dim H^k_{dR}(\Sigma) = \chi(\Sigma)$.

By Gauss-Bonnet, that is also $\frac{1}{2\pi}\int_\Sigma \underbrace{K_G}_{\text{Gaussian curvature}}\, \mathrm{vol}_g = -\frac{1}{2\pi i}\int_\Sigma F$

where $F \in \Omega^2(\Sigma, \mathfrak{u}(1))$ is the curvature 2-form of any $\underset{\overset{\shortparallel}{SO(2)}}{U(1)}$-compatible connection ∇ on $T\Sigma \to \Sigma$ $\xrightarrow{\text{(Chern-Weil theory)}}$ first Chern class

$c_1(T\Sigma) := \left[-\frac{1}{2\pi i} F\right] \in H^2_{dR}(\Sigma)$. $\therefore$ $\boxed{\mathrm{ind}(d + d^*) = \int_\Sigma c_1(T\Sigma)}$

ex 2: $E \to \Sigma$ hol. V.B. of rank m over a closed Riemann surface Σ $\rightsquigarrow$

Riemann-Roch formula: $\boxed{\mathrm{ind}(\bar\partial) = \frac{m}{2}\chi(\Sigma) + \int_\Sigma c_1(E)}$

ex 3 (Hirzebruch signature thm): M closed oriented, $\dim = 4k$ $\Rightarrow$

$\exists$ nondeg. quadratic form Q on $H^{2k}_{dR}(M)$, $Q(\alpha, \beta) := \int_M \alpha \wedge \beta$.

Its signature $\sigma(M) \in \mathbb{Z}$ is also the index of an elliptic operator.

Case $k=1$: $\sigma(M) = \frac{1}{3}\int_M p_1(M)$, $\quad p_k(M) \in H^{4k}_{dR}(M)$ are the kth Pontryagin classes of $TM \to M$.

application (using homology computations & Chern classes of almost cpx mfds):
$\nexists$ symplectic form on the connected sum $\mathbb{C}P^2 \# \mathbb{C}P^2$.

general thm (Atiyah–Singer): For any even-dim. closed oriented mfd M w/ a "graded Clifford bndl" $S = S_+ \oplus S_- \to M$ & associated "Dirac operator" $D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} : \Gamma(S_+ \oplus S_-) \circlearrowleft$,

$$\mathrm{ind}(D_+) = \int_M \underbrace{\hat{A}(TM)}_{\text{"}\hat{A}\text{-genus"}} \wedge \underbrace{ch(S/\Delta)}_{\text{"relative Chern character"}}$$

Above exs. are all special cases of this.

idea of the "heat kernel" pf: Spse $A : \mathbb{C}^m \to \mathbb{C}^n$, want to compute $\mathrm{ind}(A)$.
Let $L = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} : \mathbb{C}^{m+n} \circlearrowleft$, a self-adjoint op. w/ $L^2 = \begin{pmatrix} A^*A & 0 \\ 0 & AA^* \end{pmatrix}$ a nonneg. self-adjoint op. (Compare: for Dirac ops. $D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$ such as $d + d^*$ on $\Omega^{\text{even}}(M) \oplus \Omega^{\text{odd}}(M)$, "$D^2$ = Laplacian".)

Since $\ker A^* = (\mathrm{im}\, A)^\perp$, $\mathrm{ind}(A) = \dim\ker A - \dim\ker A^*$.
But $\langle u, A^*Au\rangle = \|Au\|^2 \Rightarrow \ker A^*A = \ker A$, sim. $\ker AA^* = \ker A^*$.
Defn $\epsilon := \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ & $f(t) := \mathrm{tr}(\epsilon e^{-tL^2})$ $\forall\, t \geq 0$.

EXERCISE:

(1) $f'(t) = 0$ $\forall t$. (computation)
(2) $f(0) = \mathrm{tr}(\epsilon) = m - n$.
(3) $\lim_{t\to\infty} f(t) = \dim\ker A^*A - \dim\ker AA^* = \mathrm{ind}(A)$: indeed,

$\epsilon e^{-tL^2} = \begin{pmatrix} e^{-tA^*A} & \\ & -e^{-tAA^*} \end{pmatrix}$, & in an eigenbasis of A^*A,

$$e^{-tA^*A} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & e^{-t\lambda_1} & \\ & & & & \ddots \\ & & & & & e^{-t\lambda_r} \end{pmatrix} \xrightarrow{t\to\infty} \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \Big\}\dim\ker A^*A,$$

similarly, $\mathrm{tr}(e^{-tAA^*}) \to \dim\ker AA^*$.

main tasks:

(1) Make sense of "$e^{-t\Delta}$" for $\Delta :=$ square of a Dirac operator.
$\rightsquigarrow$ family of bdd operators on Hilbert space satisfying $(\partial_t + \Delta)e^{-t\Delta} = 0$ "heat eqn"

(2) Compute $\lim_{t\to 0} \mathrm{tr}(\epsilon e^{-t\Delta})$ in terms of topological invts from Chern–Weil theory.